

MAJORITY DYNAMICS ON SPARSE RANDOM GRAPHS

GOPAL GOEL AND ASHWIN SAH

ABSTRACT. Consider an undirected graph G on n nodes, with each node holding an opinion from one of two options. We then run *majority dynamics*: at each discrete time step, each node updates its opinion according to majority rule among its neighbors. A well-studied conjecture of Benjamini, Chan, O'Donnell, Tamuz, and Tan states that if G is drawn from the random binomial model $\mathbb{G}(n, p)$, and if the starting opinions are chosen uniformly at random, then as long as $pn \rightarrow \infty$ the network will converge to 99% consensus with high probability.

We confirm this conjecture as long as $pn \geq n^\varepsilon$ for any $\varepsilon > 0$, showing that full unanimity is reached with high probability in $O(1/\varepsilon)$ days. This result improves on a line of results by Fountoulakis, Kang, and Makai, who achieve $\varepsilon = 1/2$; Chakraborti, Kim, Lee, and Tran, who achieve $\varepsilon = 2/5$; and Jaffe, who achieves $\varepsilon = 1/3$.

The proof technique involves iteratively revealing the “opinion histories” for each node through time, keeping track of the degrees between every node and each of 2^t node clusters after step t . Our analysis of this process requires intricate analysis of degree-constrained random graph models using graph enumeration tools from the work of McKay and Wormald as well as Canfield, Greenhill, and McKay. In doing so, we must bridge between our discrete process and the evolution of certain idealized Gaussian parameter.

1. INTRODUCTION

The evolution of opinions held by a network of individuals has been studied greatly due to its key role in social systems. This process is incredibly complex, and even the understanding of natural mathematical models of opinion dynamics has seen much interest. The so-called *majority dynamics* provide one central model which has been studied in fields such as biophysics [?] and psychology [?] since the mid-1900s.

Explicitly, consider a graph G which captures a network structure between n individuals. Each node $i \in \{1, \dots, n\}$ is given an initial opinion $A_i^{(0)} \in \{\pm 1\}$ which evolves over discrete time: at each time step all nodes simultaneously adopt the majority opinion of their neighbors. Mathematically, we have $A_i^{(t+1)} = \text{sign}(\sum_{j \sim i} A_j^{(t)})$ for integers $t \geq 0$ (if the sum is 0, we set $A_i^{(t+1)} = A_i^{(t)}$). A fundamental fact following from work of Goles and Olivos [?] is that on finite graphs, this process will eventually result in a fixed point or a cycle of period 2.

Beyond this, we can see that localities within G affect the end result of majority dynamics in nontrivial ways, and much work has studied different characteristics of this interaction. Notably, in recent work Tran and Vu [?] demonstrated a surprising “power of few” phenomenon, demonstrating that in dense random networks a constant-sized discrepancy in opinion share is likely to quickly balloon to consensus. This result saw further work by Berkowitz and Devlin [?] and was sharpened to a “power of one” [?].

The first systematic study of the behavior of majority dynamics in the natural setting of the Erdős-Rényi random network $\mathbb{G}(n, p)$ was performed by Benjamini, Chan, O'Donnell, Tamuz, and Tan [?]. Amid their many striking results, they posed the following provocative conjecture.

Conjecture 1.1 (From [?, Conjecture 1.5]). *Let $pn \rightarrow \infty$, and fix any constant $\eta > 0$. We consider majority dynamics run on $\mathbb{G}(n, p)$ with uniformly random starting opinions $A_i^{(0)} \in \{\pm 1\}$ for $i \in$*

$V(G)$. Then with probability tending towards 1 as $n \rightarrow \infty$, we have near consensus $|\sum_{i \in V(G)} A_i^{(t)}| \geq (1 - 2\eta)n$ for all sufficiently large times t .

Benjamini, Chan, O'Donnell, Tamuz, and Tan [?] show that there is a constant $C > 0$ such that if $p \geq Cn^{-1/2}$ then with probability at least 0.4 at time $t = 4$ all vertices in fact unanimously hold the initial majority opinion.

Much of the intervening work on majority dynamics in the random network setting has been towards the aim of demonstrating this conjecture, or exploring variations where one side is given a large lead to start with. (Note that with high probability, a uniformly random starting opinion ensemble will result in a lead of size $\Theta(\sqrt{n})$.)

Fountoulakis, Kang, and Makai [?] improved the unanimity probability for $p \geq Cn^{-1/2}$ to $1 - \varepsilon$, where $\varepsilon = \varepsilon(C) \rightarrow 0$ as $C \rightarrow \infty$. The range of p for which unanimity occurs with probability at least $1 - \varepsilon$ was extended to $p \geq Cn^{-3/5} \log n$ by Chakraborti, Kim, Lee, and Tran [?], and then very recently to $p \geq Cn^{-2/3}$ by Jaffe [?].

We extend the range to the optimal exponent range, namely, $p \geq n^{-1+\varepsilon}$ for any $\varepsilon > 0$.

Theorem 1.2. Fix a constant $\theta \in (1/2, 1)$ and let $k = 2\lfloor \frac{1}{1-\theta} \rfloor + 3$. Sample a random graph $G \sim \mathbb{G}(n, n^{-\theta})$ and run majority dynamics on G with uniformly random starting opinions $A_i^{(0)} \in \{\pm 1\}$ for $i \in V(G)$. Then with probability tending towards 1 as $n \rightarrow \infty$, we have unanimity after k steps:

$$\left| \sum_{i \in V(G)} A_i^{(t)} \right| = n \text{ for all } t \geq k.$$

Remark. Combined with the prior work dealing with $p \geq n^{-1/2}$, this resolves [Conjecture 1.1](#) for p ranging down to the optimal exponent. Although our proof is structured to take in a constant parameter θ and handle values $p = \Theta(n^{-\theta})$, if care is taken to ensure uniform dependence on parameters then our methods can prove a uniform result over all $p \geq n^{-1+o(1)}$.

A major innovation in our proof is the ability to control the evolution of the majority dynamics process out to many time-steps into the future. Prior works [?, ?, ?]

Merge body bibliographic references into BibTeX, and also add in new citations such as 2503.14401 and SS24, GO80, Jaf25, TV23 (sparse power of few), and also update all references including CKLT21

1.1. Proof techniques. Unclean bullet points:

- As before, we consider the case where $p = n^{-\theta}$ for some fixed constant $\theta \in (0, 1)$.
- We are able to give a quantitative estimate on how the lead size grows every day. Concretely, we start the process with a lead of size $\tau\sqrt{n}$ for some constant $\tau > 0$, and we are able to show that after k steps of evolution, the lead concentrates around $\tau c_k \sqrt{n} \sqrt{pn}^k$ for some explicit universal constant $c_k > 0$ (we don't compute the exact value but we give an algorithm for computing it), as long as $\sqrt{n} \sqrt{pn}^k = o(n)$.
- For completing the process, we cite the results of [5] and complete the process from a lead of $\tilde{\Omega}(n/\sqrt{pn})$ to full consensus.

2. SETUP

2.1. Definition of Majority Dynamics. Let

$$\geq_s := \begin{cases} \geq & \text{if } s = 00 \\ < & \text{if } s = 01 \\ > & \text{if } s = 10 \\ \leq & \text{if } s = 11. \end{cases}$$

View strings in $\{0, 1\}^k$ as being 0-indexed, so if $s \in \{0, 1\}^k$, then $s = s_0 s_1 \cdots s_{k-1}$, where $s_i \in \{0, 1\}$.

For any $x \in \mathbb{R}^{\{0,1\}^k}$ and $s \in \{0, 1\}^k$, let $\mathcal{I}_s(x)$ denote the assertion that

$$\sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} x_t \geq_{s_{r-1} s_r} 0$$

for all $1 \leq r \leq k-1$. For any $x \in \mathbb{R}^{\{0,1\}^k}$ and $s \in \{0, 1\}^{k+1}$, let $\mathcal{J}_s(x)$ denote the assertion that

$$\sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} x_t \geq_{s_{r-1} s_r} 0$$

for all $1 \leq r \leq k$.

Call a graph G *politicized* if it is endowed with a vertex partition $V(G) = V_0(G) \sqcup V_1(G)$, which we also refer to as the *day-1 partition*. For any politicized graph G and for every positive integer k , we define its *day-k partition* as $V(G) = \sqcup_{s \in \{0,1\}^k} V_s(G)$, where the parts are defined recursively through

$$V_{sb}(G) := \left\{ x \in V_s(G) : \mathcal{J}_{sb} \left(\left(\deg_{V_t(G)} x \right)_{t \in \{0,1\}^k} \right) \right\},$$

where $s \in \{0, 1\}^k$ and $b \in \{0, 1\}$. It follows from this construction that $\mathcal{I}_s \left(\left(\deg_{V_t(G)} x \right)_{t \in \{0,1\}^k} \right)$ is true for all $x \in V_s(G)$.

2.2. Degree Vectors. For any partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$, define

$$\mathcal{D}^{\{V_s\}_{s \in \{0,1\}^k}} := \left\{ \mathbf{d} \in \left(\mathbb{Z}^{\{0,1\}^k} \right)^V : \mathbf{d}_t^{(x)} \in [0, |V_t| - \mathbf{1}_{x \in V_t}] \ \forall t \wedge \sum_{x \in V_s} \mathbf{d}_t^{(x)} = \sum_{x \in V_t} \mathbf{d}_s^{(x)} \ \forall s \neq t \wedge \sum_{x \in V_s} \mathbf{d}_s^{(x)} \in 2\mathbb{Z} \ \forall s \right\}$$

and

$$\mathcal{M}^{\{V_s\}_{s \in \{0,1\}^k}} := \left\{ m \in \mathbb{Z}^{\{0,1\}^k \times \{0,1\}^k} : m_{s,t} = m_{t,s} \wedge m_{s,s} \in 2\mathbb{Z} \wedge m_{s,t} \in [0, |V_s|(|V_t| - \mathbf{1}_{s=t})] \ \forall s, t \right\}.$$

For any $\mathbf{d} \in \mathcal{D}^{\{V_s\}_{s \in \{0,1\}^k}}$, let $m(\mathbf{d}) \in \mathcal{M}^{\{V_s\}_{s \in \{0,1\}^k}}$ be given by $m(\mathbf{d})_{s,t} = \sum_{x \in V_s} \mathbf{d}_t^{(x)}$. For any graph G on partitioned vertex set $V = \sqcup_{s \in \{0,1\}^k} V_s$, let $\delta_{\{V_s\}_{s \in \{0,1\}^k}}(G) \in \mathcal{D}^{\{V_s(G)\}_{s \in \{0,1\}^k}}$ be given by $\delta_{\{V_s\}_{s \in \{0,1\}^k}}(G)_t^{(x)} = \deg_{V_t} x$. Note that $m \left(\delta_{\{V_s\}_{s \in \{0,1\}^k}}(G) \right)$ gives the edge counts between the parts of the partition. For politicized graphs G , we further define $\delta_k(G) = \delta_{\{V_s(G)\}_{s \in \{0,1\}^k}}(G)$.

Consider a partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$, and a degree vector $\mathbf{d} \in \mathcal{D}^{\{V_s\}_{s \in \{0,1\}^k}}$. Let $\mathcal{I}(\mathbf{d})$ denote the collective inequality conditions $\mathcal{I}_s(\mathbf{d}^{(x)})$ for all $s \in \{0, 1\}^k$ and $x \in V_s$. For a probability $p \in (0, 1)$, let $\kappa_p(\mathbf{d})$ denote the conditions

$$|\mathbf{d}_t^{(x)} - p|V_t|| \leq (p|V|)^{4/7}$$

for all $t \in \{0, 1\}^k$ and $x \in V$. For any probability $p \in (0, 1)$ and constant $C > 0$, let $\Gamma_{C,p}(\mathbf{d})$ denote the conditions

$$\frac{1}{p|V|^2} \sum_{x \in V_s} \left(\mathbf{d}_t^{(x)} - \frac{m(\mathbf{d})_{s,t}}{|V_s|} \right)^2 \leq C$$

for all $s, t \in \{0, 1\}^k$. Define

$$\mathcal{D}_{\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} = \left\{ \mathbf{d} \in \mathcal{D}^{\{V_s\}_{s \in \{0,1\}^k}} : \mathcal{I}(\mathbf{d}) \right\},$$

and define $\mathcal{D}_{\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}}$ and $\mathcal{D}_{\Gamma_{C,p}}^{\{V_s\}_{s \in \{0,1\}^k}}$ similarly. Furthermore, for any $m \in \mathcal{M}^{\{V_s\}_{s \in \{0,1\}^k}}$ define

$$\mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} = \left\{ \mathbf{d} \in \mathcal{D}^{\{V_s\}_{s \in \{0,1\}^k}} : m(\mathbf{d}) = m \right\}.$$

For any collection of symbols Π , define

$$\mathcal{D}_\Pi^{\{V_s\}_{s \in \{0,1\}^k}} = \bigcap_{\pi \in \Pi} \mathcal{D}_\pi^{\{V_s\}_{s \in \{0,1\}^k}}.$$

2.3. Probability Distributions on Degree Vectors. Let $\mathbb{P}_1^{\{V_s\}_{s \in \{0,1\}^k}}$ denote the probability distribution on $\mathcal{D}^{\{V_s\}_{s \in \{0,1\}^k}}$ given by sampling $G \sim \mathbb{G}(V, p)$, and then outputting $\delta_{\{V_s\}_{s \in \{0,1\}^k}}(G)$.

For any collection of symbols Π , let $\mathbb{P}_{1,\Pi}^{\{V_s\}_{s \in \{0,1\}^k}}$ denote $\mathbb{P}_1^{\{V_s\}_{s \in \{0,1\}^k}}$ conditioned on the output being in $\mathcal{D}_\Pi^{\{V_s\}_{s \in \{0,1\}^k}}$. We usually view $\mathbb{P}_{1,\Pi}^{\{V_s\}_{s \in \{0,1\}^k}}$ as a probability distribution on $\mathcal{D}_\Pi^{\{V_s\}_{s \in \{0,1\}^k}}$, but we may also view it as a probability distribution on $\mathcal{D}_{\Pi'}^{\{V_s\}_{s \in \{0,1\}^k}}$, where $\Pi' \subset \Pi$, and $\mathbb{P}_{1,\Pi}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathbf{d}] = 0$ if $\mathbf{d} \notin \mathcal{D}_\Pi^{\{V_s\}_{s \in \{0,1\}^k}}$.

For any set of weights $q \in (0, 1)^{\{0,1\}^k \times \{0,1\}^k}$, let $\mathbb{P}_{2,q}^{\{V_s\}_{s \in \{0,1\}^k}}$ be given by sampling $\mathbf{d}_t^{(x)} \sim \text{Bin}(n_t - \mathbf{1}_{s=t}, q_{s,t})$ independently for all $s, t \in \{0,1\}^k$, $x \in V_s$. Again, for any collection of symbols Π , let $\mathbb{P}_{2,q,\Pi}^{\{V_s\}_{s \in \{0,1\}^k}}$ denote $\mathbb{P}_{2,q}^{\{V_s\}_{s \in \{0,1\}^k}}$ conditioned on the output being in $\mathcal{D}_\Pi^{\{V_s\}_{s \in \{0,1\}^k}}$. If $m \in \Pi$ for some $m \in \mathcal{M}^{\{V_s\}_{s \in \{0,1\}^k}}$, then the distribution doesn't depend on the value of q , so we drop q from the subscript.

2.4. Technical Size Conditions. Fix a constant $T > 1$ and a probability $p \in (0, 1)$. Call a partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$ (T, p) -tolerant if $|V_s| \geq T^{-1}|V|$ for all $s \in \{0,1\}^k$ and

$$\left| \sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} |V_t| \right| \leq T \frac{|V|}{\sqrt{p|V|}}.$$

For a (T, p) -tolerant partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$, call an element $m \in \mathcal{M}^{\{V_s\}_{s \in \{0,1\}^k}}(T, p)$ -tolerant if

$$\sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} m_{s,t} \geq_{s_{r-1}s_r} (-1)^{s_r} T^{-1} \frac{|V|^2 p}{\sqrt{|V|p}}$$

for all $s \in \{0,1\}^k$ and $1 \leq r \leq k-1$, and

$$|m_{s,t} - p|V_s||V_t|| \leq T \frac{|V|^2 p}{\sqrt{|V|p}}$$

for all $s, t \in \{0,1\}^k$. Let $\mathcal{M}_{T,p}^{\{V_s\}_{s \in \{0,1\}^k}}$ denote the subset of (T, p) -tolerant elements of $\mathcal{M}^{\{V_s\}_{s \in \{0,1\}^k}}$.

Finally, we introduce the notion of *solvability* for a T -tolerant partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$ and an associated $m \in \mathcal{M}_{T,p}^{\{V_s\}_{s \in \{0,1\}^k}}$. In particular, we say m is solvable with respect to the partitioned set with solution $q \in (0, 1)^{\{0,1\}^k \times \{0,1\}^k}$ if $|q_{s,t} - p| \leq Tp/\sqrt{p|V|}$ and $\mathbb{E}[A_{s,t} | \mathcal{I}_s(A_s, \bullet)] = m_{s,t}/|V_s|$, where we sample $A_{s,t} \sim \text{Bin}(|V_t| - \mathbf{1}_{s=t}, q_{s,t})$ independently.

3. TRANSFERENCE

The key result of this section is as follows.

Theorem 3.1. *Fix a real parameter $\theta \in (1/2, 1)$, a constant $T > 1$, and a decreasing function $f : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$ satisfying $\lim_{N \rightarrow \infty} f(N) = 0$. For all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$, the following is true.*

Consider a (T, p) -tolerant partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$ with $|V| = N$, and an associated solvable $m \in \mathcal{M}_{T,p}^{\{V_s\}_{s \in \{0,1\}^k}}$. For any event $\mathcal{E} \subset \mathcal{D}^{\{V_s\}_{s \in \{0,1\}^k}}$ satisfying $\mathbb{P}_{2,m,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathcal{E}] \leq f(N)$, we have $\mathbb{P}_{1,m,\mathcal{I},\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathcal{E}] = 1 - o(1)$.

The key tool in this proof is the graph enumeration theorems of Wormald. We elect to state the key lemmas we need, and defer the main content of the proofs to [Section C](#).

Lemma 3.2. *Fix a real parameter $\theta \in (1/2, 1)$, and constants $T > 1$, $C > 0$. There exists a constant $C' > 1$ such that for all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$, the following is true.*

Consider a (T, p) -tolerant partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$ with $|V| = N$ and an associated $m \in \mathcal{M}_{T,p}^{\{V_s\}_{s \in \{0,1\}^k}}$. Then,

$$\frac{1}{C'} \mathbb{P}_{2,m}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathbf{d}] \leq \mathbb{P}_{1,m}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathbf{d}] \leq C' \mathbb{P}_{2,m}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathbf{d}]$$

for all $\mathbf{d} \in \mathcal{D}_{m,\kappa_p,\Gamma_{C,p}}^{\{V_s\}_{s \in \{0,1\}^k}}$.

Proof. Adopt the definitions [Definition C.1](#) and [Definition C.2](#). Label all the vertices of the graph as

$$V_s = \{x_{s,1}, \dots, x_{s,|V_s|}\}.$$

There is a natural isomorphism

$$\mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \rightarrow \prod_{s \in \{0,1\}^k} \mathcal{D}_{|V_s|,m_{s,s}/2}^{\text{graph}} \times \prod_{\substack{s,t \in \{0,1\}^k \\ s \neq t}} \mathcal{D}_{|V_s|,|V_t|,m_{s,t}}^{\text{bigraph}}$$

given by

$$\mathbf{d} \mapsto \left(\left((\mathbf{d}_s^{x_{s,i}})_{i \in [|V_s|]} \right)_{s \in \{0,1\}^k}, \left(\left((\mathbf{d}_t^{x_{s,i}})_{i \in [|V_s|]}, (\mathbf{d}_s^{x_{t,j}})_{j \in [|V_t|]} \right)_{s,t \in \{0,1\}^k, s \neq t} \right).$$

The various probability distributions relate under this isomorphism through the relations

$$\begin{aligned} \mathbb{P}_{1,m}^{\{V_s\}_{s \in \{0,1\}^k}} &= \prod_{s \in \{0,1\}^k} \mathbb{P}_{G(|V_s|,m_{s,s}/2)} \times \prod_{\substack{s,t \in \{0,1\}^k \\ s \neq t}} \mathbb{P}_{G(|V_s|,|V_t|,m_{s,t})} \\ \mathbb{P}_{2,m}^{\{V_s\}_{s \in \{0,1\}^k}} &= \prod_{s \in \{0,1\}^k} \mathbb{P}_{B(|V_s|,m_{s,s}/2)} \times \prod_{\substack{s,t \in \{0,1\}^k \\ s \neq t}} \mathbb{P}_{B(|V_s|,|V_t|,m_{s,t})}. \end{aligned}$$

Note that for $\mathbf{d} \in \mathcal{D}_{m,\kappa_p,\Gamma_{C,p}}^{\{V_s\}_{s \in \{0,1\}^k}}$, the induced degree sequence $(\mathbf{d}_s^{x_{s,i}})_{i \in [|V_s|]} \in \mathcal{D}_{|V_s|,m_{s,s}/2}$ satisfies the conditions of [Theorem C.3](#) with $\alpha = 7/12$, and furthermore $\left| \gamma_2 \left((\mathbf{d}_s^{x_{s,i}})_{i \in [|V_s|]} \right) \right| = O(p)$. Therefore,

by [Theorem C.3](#), we see that

$$\log \left| \frac{\mathbb{P}_{G(|V_s|, m_{s,s}/2)} \left[(\mathbf{d}_s^{x_{s,i}})_{i \in [|V_s|]} \right]}{\mathbb{P}_{B(|V_s|, m_{s,s}/2)} \left[(\mathbf{d}_s^{x_{s,i}})_{i \in [|V_s|]} \right]} \right| = O(1).$$

We can similarly show using [Theorem C.4](#) that

$$\log \left| \frac{\mathbb{P}_{G(|V_s|, |V_t|, m_{s,t})} \left[\left((\mathbf{d}_t^{x_{s,i}})_{i \in [|V_s|]}, (\mathbf{d}_s^{x_{t,j}})_{j \in [|V_t|]} \right) \right]}{\mathbb{P}_{B(|V_s|, |V_t|, m_{s,t})} \left[\left((\mathbf{d}_t^{x_{s,i}})_{i \in [|V_s|]}, (\mathbf{d}_s^{x_{t,j}})_{j \in [|V_t|]} \right) \right]} \right| = O(1).$$

Adding together all these estimates completes the proof. $\square$

Lemma 3.3. *Fix a real parameter $\theta \in (1/2, 1)$, and a constant $T > 1$. For all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$, the following is true.*

Consider a (T, p) -tolerant partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$ with $|V| = N$ and an associated $m \in \mathcal{M}_{T,p}^{\{V_s\}_{s \in \{0,1\}^k}}$. Then,

$$\log \left| \frac{\mathbb{P}_{1,m}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathbf{d}]}{\mathbb{P}_{2,m}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathbf{d}]} \right| = O((pN)^{2/7})$$

for all $\mathbf{d} \in \mathcal{D}_{m, \kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}}$.

Proof. The proof is the exact same as the proof of [Lemma 3.2](#), except that our γ_2 bound (and the corresponding σ^2 bigraph bound) is weakened to $\left| \gamma_2 \left((\mathbf{d}_s^{x_{s,i}})_{i \in [|V_s|]} \right) \right| = O((pN)^{1/7})$, which we derive directly from $\kappa_p(\mathbf{d})$. In this weaker setting, [Theorem C.3](#) and [Theorem C.4](#) give

$$\log \left| \frac{\mathbb{P}_{G(|V_s|, m_{s,s}/2)} \left[(\mathbf{d}_s^{x_{s,i}})_{i \in [|V_s|]} \right]}{\mathbb{P}_{B(|V_s|, m_{s,s}/2)} \left[(\mathbf{d}_s^{x_{s,i}})_{i \in [|V_s|]} \right]} \right| = O((pN)^{2/7})$$

and

$$\log \left| \frac{\mathbb{P}_{G(|V_s|, |V_t|, m_{s,t})} \left[\left((\mathbf{d}_t^{x_{s,i}})_{i \in [|V_s|]}, (\mathbf{d}_s^{x_{t,j}})_{j \in [|V_t|]} \right) \right]}{\mathbb{P}_{B(|V_s|, |V_t|, m_{s,t})} \left[\left((\mathbf{d}_t^{x_{s,i}})_{i \in [|V_s|]}, (\mathbf{d}_s^{x_{t,j}})_{j \in [|V_t|]} \right) \right]} \right| = O((pN)^{2/7})$$

respectively. Adding these all together gives the desired bound. $\square$

Lemma 3.4. *Fix a real parameter $\theta \in (1/2, 1)$, and a constant $T > 1$. There exists a constant $C > 0$ such that for all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$, the following is true.*

Consider a (T, p) -tolerant partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$ with $|V| = N$ and an associated $m \in \mathcal{M}_{T,p}^{\{V_s\}_{s \in \{0,1\}^k}}$. Then,

$$\mathbb{P}_{1,m,\mathcal{I},\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{D}_{\Gamma_{C,p}}^{\{V_s\}_{s \in \{0,1\}^k}} \right] \geq 1 - e^{-N}.$$

Proof. Drop all superscripts with the vertex partition for ease of notation, and let $n_s = |V_s|$ for all $s \in \{0,1\}^k$. The proof follows that of Lemma 5.3 of [\[4\]](#).

We begin by lower bounding $\mathbb{P}_1[\mathcal{D}_m]$. Indeed, we find that

$$\mathbb{P}_1[\mathcal{D}_m] = \prod_{s \in \{0,1\}^k} \mathbb{P}[\text{Bin}(n_s(n_s - 1)/2, p) = m_{s,s}/2] \prod_{\substack{s,t \in \{0,1\}^k \\ s \neq t}} \mathbb{P}[\text{Bin}(n_s n_t, p) = m_{s,t}].$$

Fixing $s, t \in \{0, 1\}^k$ and letting $R = n_s(n_t - \mathbf{1}_{s=t})/(1 + \mathbf{1}_{s=t})$ and $r = m_{s,t} - \lfloor Rp \rfloor$, we find that there are constants $C_1, C_2 > 0$ such that

$$\begin{aligned} \mathbb{P}[\text{Bin}(R, p) = \lfloor Rp \rfloor + r] &= \mathbb{P}[\text{Bin}(R, p) = \lfloor Rp \rfloor] \prod_{j=1}^r \frac{1 - \frac{r-j}{R - \lfloor Rp \rfloor}}{1 + \frac{j}{\lfloor Rp \rfloor}} \\ &\geq \exp(-C_1 r^2 / Rp) \\ &\geq \exp(-C_2 N), \end{aligned}$$

so by multiplying all these estimates together, we find that there is a constant $C_3 > 0$ such that

$$\mathbb{P}_1[\mathcal{D}_m] \geq \exp(-C_3 N).$$

Next, we provide a similar lower bound on $\mathbb{P}_{1,m}[\mathcal{D}_{m,\mathcal{I},\kappa_p,\Gamma_{C,p}}]$. Note that the event

$$\mathcal{E} := \left\{ \mathbf{d} \in \mathcal{D}_m : \mathbf{d}_t^{(x)} \in \left[\frac{m_{s,t}}{n_s} - \frac{0.01}{T2^k} \sqrt{Np}, \frac{m_{s,t}}{n_s} + \frac{0.01}{T2^k} \sqrt{Np} \right] \text{ for all } s, t \in \{0, 1\}^k, x \in V_s \right\}$$

is a subset of $\mathcal{D}_{m,\mathcal{I},\kappa_p,\Gamma_{C,p}}$, so it suffices to lower bound $\mathbb{P}_{1,m}[\mathcal{E}]$. Choose $q \in (0, 1)^{\{0,1\}^k \times \{0,1\}^k}$ to be $q_{s,t} = p$ for all $s, t \in \{0, 1\}^k$. There is a constant $C_4 > 0$ such that for $\mathbf{d} \in \mathcal{E}$,

$$\mathbb{P}_{2,m}[\mathbf{d}] \geq \mathbb{P}_{2,q}[\mathbf{d}] \geq \left(C_4 / \sqrt{pN} \right)^{2^k N}.$$

Thus, by Lemma 3.2, we find that there is a constant $C_5 > 0$ such that

$$\mathbb{P}_{1,m}[\mathbf{d}] \geq \sqrt{pN}^{-2^k N} \exp(-C_5 N).$$

By Lemma C.6, we find that there is some constant $C_6 > 0$ such that

$$|\mathcal{E}| \geq \left(\sqrt{pN} \right)^{2^k N} \exp(-C_6 N),$$

so

$$\mathbb{P}_{1,m}[\mathcal{D}_{m,\mathcal{I},\kappa_p,\Gamma_{C,p}}] \geq \mathbb{P}_{1,m}[\mathcal{E}] \geq \exp(-(C_5 + C_6)N).$$

Combining these two estimates, we find that

$$\mathbb{P}_1[\mathcal{D}_{m,\mathcal{I},\kappa_p,\Gamma_{C,p}}] \geq \exp(-(C_3 + C_5 + C_6)N) = \exp(-C_7 N).$$

Thus,

$$\mathbb{P}_{1,m,\mathcal{I},\kappa_p} \left[\mathcal{D}_{\bar{\Gamma}_{C,p}} \right] = \frac{\mathbb{P}_1 \left[\mathcal{D}_{m,\mathcal{I},\kappa,\bar{\Gamma}_{C,p}} \right]}{\mathbb{P}_1 \left[\mathcal{D}_{m,\mathcal{I},\kappa_p} \right]} \leq \exp(C_7 N) \mathbb{P}_1 \left[\mathcal{D}_{\bar{\Gamma}_{C,p}} \right].$$

Note that Lemma C.5 directly implies that there exists a constant $C > 0$ such that

$$\mathbb{P}_1 \left[\mathcal{D}_{\bar{\Gamma}_{C,p}} \right] \leq \exp(-(2C_7 + 1)N),$$

so

$$\mathbb{P}_{1,m,\mathcal{I},\kappa_p} \left[\mathcal{D}_{\bar{\Gamma}_{C,p}} \right] \leq \exp(-N),$$

as desired. $\square$

Lemma 3.5. Fix a real parameter $\theta \in (1/2, 1)$, and a constant $T > 1$. For all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$, the following is true.

Consider a (T, p) -tolerant partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$ with $|V| = N$ and an associated solvable $m \in \mathcal{M}_{T,p}^{\{V_s\}_{s \in \{0,1\}^k}}$ with solution $q \in (0, 1)^{\{0,1\}^k \times \{0,1\}^k}$. Then,

$$\mathbb{P}_{2,q,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{D}_{\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}} \right] = 1 - O(N^{-\omega(1)})$$

and

$$\mathbb{P}_{2,m,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{D}_{\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}} \right] = 1 - O(N^{-\omega(1)}).$$

Proof. We begin by showing the first equality. Since there are a polynomial number of (t, x) pairs, it suffices to show that

$$\mathbb{P} \left[|A_{s,t} - p|V_t|| > (pN)^{4/7} \middle| \mathcal{I}_s(A_{s,\bullet}) \right] = O(N^{-\omega(1)}),$$

where we sample $A_{s,t} \sim \text{Bin}(|V_t| - \mathbf{1}_{s=t}, q_{s,t})$ all independently. We have $\mathbb{P}[\mathcal{I}_s(A_{s,\bullet})] = \Omega(1)$ by [Theorem A.2](#), so it suffices to prove

$$\mathbb{P} \left[|A_{s,t} - p|V_t|| > (pN)^{4/7} \middle| \mathcal{I}_s(A_{s,\bullet}) \right] = O(N^{-\omega(1)}),$$

which is true by a trivial Chernoff bound.

In order to show the second equality, note that by [Theorem A.3](#), we have that

$$\mathbb{P}_{2,q,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \right] = \Omega \left((N^2 p)^{-2^{2k-1}} \right).$$

Therefore,

$$\mathbb{P}_{2,m,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{D}_{\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}} \right] = \frac{\mathbb{P}_{2,q,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{D}_{\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}} \cap \mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \right]}{\mathbb{P}_{2,q,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \right]} = O(N^{-\omega(1)}),$$

as desired. $\square$

We now prove the main result of this section.

Proof of [Theorem 3.1](#). We drop all superscripts of the partitioned set for legibility.

Let C be the constant in [Lemma 3.4](#), and let C' be the associated constant from [Lemma 3.2](#). Note that

$$\begin{aligned} \mathbb{P}_{1,m,\mathcal{I},\kappa_p}[\bar{\mathcal{E}}] &= \frac{\sum_{\substack{\mathbf{d} \in \mathcal{D}_{m,\mathcal{I},\kappa_p,\Gamma_{C,p}} \\ \mathbf{d} \notin \mathcal{E}}} \mathbb{P}_{1,m}[\mathbf{d}] + \sum_{\substack{\mathbf{d} \in \mathcal{D}_{m,\mathcal{I},\kappa_p,\bar{\Gamma}_C} \\ \mathbf{d} \notin \mathcal{E}}} \mathbb{P}_{1,m}[\mathbf{d}]}{\sum_{\mathbf{d} \in \mathcal{D}_{m,\mathcal{I},\kappa_p,\Gamma_{C,p}}} \mathbb{P}_{1,m}[\mathbf{d}] + \sum_{\mathbf{d} \in \mathcal{D}_{m,\mathcal{I},\kappa_p,\bar{\Gamma}_C}} \mathbb{P}_{1,m}[\mathbf{d}]} \\ &\leq (1 + o(1)) \frac{\sum_{\substack{\mathbf{d} \in \mathcal{D}_{m,\mathcal{I},\kappa_p,\Gamma_{C,p}} \\ \mathbf{d} \notin \mathcal{E}}} \mathbb{P}_{1,m}[\mathbf{d}]}{\sum_{\mathbf{d} \in \mathcal{D}_{m,\mathcal{I},\kappa_p,\Gamma_{C,p}}} \mathbb{P}_{1,m}[\mathbf{d}]} + o(1) \\ &\leq 2C'^2 \mathbb{P}_{2,m,\mathcal{I},\kappa_p,\Gamma_{C,p}}[\bar{\mathcal{E}}] + o(1), \end{aligned}$$

where the first inequality follows from [Lemma 3.4](#) and the second inequality follows from [Lemma 3.2](#).

Thus, it suffices to show that $\mathbb{P}_{2,m,\mathcal{I},\kappa_p,\Gamma_{C,p}}[\mathcal{E}] = 1 - o(1)$.

Begin by noting that

$$\mathbb{P}_{2,m,\mathcal{I},\kappa_p}[\mathcal{D}_{\bar{\Gamma}_C}] = \frac{\sum_{\mathbf{d} \in \mathcal{D}_{m,\mathcal{I},\kappa_p,\bar{\Gamma}_C}} \mathbb{P}_{2,m}[\mathbf{d}]}{\sum_{\mathbf{d} \in \mathcal{D}_{m,\mathcal{I},\kappa_p}} \mathbb{P}_{2,m}[\mathbf{d}]}$$

$$\begin{aligned}
&= \exp(O((pN)^{2/7})) \frac{\sum_{\mathbf{d} \in \mathcal{D}_{m, \mathcal{I}, \kappa_p, \bar{\Gamma}_C}} \mathbb{P}_{1,m}[\mathbf{d}]}{\sum_{\mathbf{d} \in \mathcal{D}_{m, \mathcal{I}, \kappa_p}} \mathbb{P}_{1,m}[\mathbf{d}]} \\
&= \exp(O((pN)^{2/7})) \mathbb{P}_{1,m, \mathcal{I}, \kappa_p}[\mathcal{D}_{\bar{\Gamma}_C}] \\
&= o(1),
\end{aligned}$$

where the second equality follows from [Lemma 3.3](#), and the final equality follows from [Lemma 3.4](#).

We now see that

$$\begin{aligned}
\mathbb{P}_{2,m, \mathcal{I}, \kappa_p, \Gamma_{C,p}}[\mathcal{E}] &= \frac{\sum_{\substack{\mathbf{d} \in \mathcal{D}_{m, \mathcal{I}, \kappa_p, \Gamma_{C,p}} \\ \mathbf{d} \in \mathcal{E}}} \mathbb{P}_{2,m}[\mathbf{d}]}{\sum_{\mathbf{d} \in \mathcal{D}_{m, \mathcal{I}, \kappa_p, \Gamma_{C,p}}} \mathbb{P}_{2,m}[\mathbf{d}]} \geq \frac{\sum_{\substack{\mathbf{d} \in \mathcal{D}_{m, \mathcal{I}} \\ \mathbf{d} \in \mathcal{E}}} \mathbb{P}_{2,m}[\mathbf{d}] - \sum_{\mathbf{d} \in \mathcal{D}_{m, \mathcal{I}, \kappa_p}} \mathbb{P}_{2,m}[\mathbf{d}] - \sum_{\mathbf{d} \in \mathcal{D}_{m, \mathcal{I}, \kappa_p, \bar{\Gamma}_C}} \mathbb{P}_{2,m}[\mathbf{d}]}{\sum_{\mathbf{d} \in \mathcal{D}_{m, \mathcal{I}, \kappa_p}} \mathbb{P}_{2,m}[\mathbf{d}]} \\
&\geq \mathbb{P}_{2,m, \mathcal{I}}[\mathcal{E}] - \mathbb{P}_{2,m, \mathcal{I}}[\mathcal{D}_{\kappa_p}] - \mathbb{P}_{2,m, \mathcal{I}, \kappa_p}[\mathcal{D}_{\bar{\Gamma}_C}] \\
&= 1 - o(1),
\end{aligned}$$

where the final equality follows from [Lemma 3.5](#) and the result above. This completes the proof. $\square$

4. PART SIZE EVOLUTION

Adopt the notation from [Definition B.7](#), [Definition B.8](#), [Theorem B.11](#), and [Theorem B.12](#). For any $T > 1$, $p \in (0, 1)$, and $\delta, \tau > 0$, a pair $(V = \sqcup_{s \in \{0,1\}^k} V_s, m)$ is said to be (T, p, δ, τ) -faithful if $m \in \mathcal{M}^{\{V_s\}_{s \in \{0,1\}^k}}$, and

$$| |V_s| - \tilde{n}_s(N) - \tau \sqrt{N} \sqrt{pN}^{k-1} \varepsilon_s | \leq T \sqrt{N} \sqrt{pN}^{k-1} N^{-\delta}$$

and

$$\left| m_{s,t} - \tilde{m}_{s,t}(N) \left[1 + \tau \frac{\sqrt{pN}^{k-1}}{\sqrt{N}} \left(\frac{\varepsilon_s}{\nu_s} + \frac{\varepsilon_t}{\nu_t} \right) \right] \right| \leq T \frac{\sqrt{pN}^{k-1}}{\sqrt{N}} N^{-\delta} N^2 p,$$

where $N = |V|$.

The following main result tracks the part size evolution.

Theorem 4.1. *Fix a real parameter $\theta \in (1/2, 1)$ and constants $T, \mathcal{T} > 1$. There exists constants $T' > 1$ and $\delta > 0$ such that for all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$ and $\tau \in [\mathcal{T}^{-1}, \mathcal{T}]$, the following is true.*

Consider a partitioned set $V = V_0 \sqcup V_1$ with $|V| = N$ and $|V_0| = \lfloor N/2 \rfloor + \lfloor \tau \sqrt{N} \rfloor$. Sample a graph $G \sim \mathbb{G}(V, p)$, and politicize it according to $V_0(G) = V_0$ and $V_1(G) = V_1$. With probability $1 - o(1)$, for all positive integers $k \in [1, \lceil \frac{1}{1-\theta} \rceil]$, the pair $(V = \sqcup_{s \in \{0,1\}^k} V_s(G), m(\delta_k(G)))$ is (T', p, δ, τ) -faithful, and $\delta_k(G) \in \mathcal{D}_{\kappa_p}^{\{V_s(G)\}_{s \in \{0,1\}^k}}$.

Additionally, if $k = \frac{1}{1-\theta} + 1 \in \mathbb{Z}$, then for all $s \in \{0, 1\}^k$, we have $|V_s| \geq \overline{s_{k-1} s_{k-1}} N (\nu_s + (-1)^{s_{k-1}} \zeta_{k-1, \tau})$.

We begin by proving the induction step of this theorem with the following lemmas.

For any partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$ and $\mathbf{d} \in \mathcal{D}^{\{V_s\}_{s \in \{0,1\}^k}}$, let $\mathbb{P}_{\mathbf{d}}$ denote the uniform probability distribution on graphs G on vertex set V constrained to satisfy $\delta_{\{V_s\}_{s \in \{0,1\}^k}}(G) = \mathbf{d}$. Furthermore,

for $\mathbf{d} \in \mathcal{D}_{\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}}$ and $sb \in \{0, 1\}^{k+1}$, define

$$W_{sb}(\mathbf{d}) := \{x \in V_s : \mathcal{J}_{sb}(\mathbf{d}^{(x)})\}.$$

Lemma 4.2. Fix a real parameter $\theta \in (1/2, 1)$ and constants $T, \mathcal{T} > 1$, $\delta > 0$, and integer $k \in [1, \frac{1}{1-\theta})$. For all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$ and $\tau \in [\mathcal{T}^{-1}, \mathcal{T}]$, the following is true.

Consider a (T, p, δ, τ) -faithful pair $(V = \sqcup_{s \in \{0,1\}^k} V_s, m)$ with $|V| = N$. Then, m is solvable with respect to the partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$, and there exists a solution $q \in (0, 1)^{\{0,1\}^k \times \{0,1\}^k}$ such that the following is true. Let

$$\begin{aligned} \mathcal{E}_{\text{degrees}} &:= \left\{ \mathbf{d} \in \mathcal{D}_{\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} : \left| \mathbf{d}_t^{(x)} - p|V_t| \right| > \sqrt{pN} \frac{\log N}{\log \log N} \text{ for some } x \in V, t \in \{0,1\}^k \right\} \\ \mathcal{E}_{\text{partsize}} &:= \left\{ \mathbf{d} \in \mathcal{D}_{\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} : ||W_{sb}(\mathbf{d})| - |V_s| \mathbb{P}[\mathcal{J}_{sb}(A_{s,\bullet})|\mathcal{I}_s(A_{s,\bullet})]| > \sqrt{N} \log N \text{ for some } sb \in \{0,1\}^{k+1} \right\} \\ \mathcal{E}_{\text{edgecount}} &:= \left\{ \mathbf{d} \in \mathcal{D}_{\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} : \left| \left(\sum_{x \in W_{sb}(\mathbf{d})} \mathbf{d}_t^{(x)} \right) - |V_s| \mathbb{E}[A_{s,t} \mathbf{1}_{\mathcal{J}_{sb}(A_{s,\bullet})} | \mathcal{I}_s(A_{s,\bullet})] \right| > \frac{N^2 p}{\sqrt{N}} \log N \right. \\ &\quad \left. \text{for some } sb \in \{0,1\}^{k+1}, t \in \{0,1\}^k \right\}, \end{aligned}$$

where we sample $A_{s,t} \sim \text{Bin}(|V_t| - \mathbf{1}_{s=t}, q_{s,t})$ all independently. Then,

$$\mathbb{P}_{1,m,\mathcal{I},\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}} [\mathcal{E}_{\text{degrees}} \vee \mathcal{E}_{\text{partsize}} \vee \mathcal{E}_{\text{edgecount}}] = o(1).$$

Proof. Note that [Theorem B.12](#) implies that there is some constant T' such that $V = \sqcup_{s \in \{0,1\}^k} V_s$ is (T', p) -tolerant, and $m \in \mathcal{M}_{T',p}^{\{V_s\}_{s \in \{0,1\}^k}}$ is solvable with respect to the partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$. Let $q \in (0, 1)^{\{0,1\}^k \times \{0,1\}^k}$ be the solution specified by [Theorem B.12](#).

By an analogous argument to the proof of [Lemma 3.5](#), we find that $\mathbb{P}_{2,q,\mathcal{I}}[\mathcal{E}_{\text{degrees}}] = O(N^{-\omega(1)})$. If we sample $\mathbf{d} \sim \mathbb{P}_{2,q,\mathcal{I}}$, we see that

$$|W_{sb}(\mathbf{d})| \sim \text{Bin}(|V_s|, \mathbb{P}[\mathcal{J}_{sb}(A_{s,\bullet})|\mathcal{I}_s(A_{s,\bullet})])$$

where we sample $A_{s,t} \sim \text{Bin}(|V_t| - \mathbf{1}_{s=t}, q_{s,t})$ independently. By [Theorem B.12](#), there exists a constant $\phi \in (0, 1/2)$ such that

$$\mathbb{P}[\mathcal{J}_{sb}(A_{s,\bullet})|\mathcal{I}_s(A_{s,\bullet})] \in (\phi, 1 - \phi)$$

for all N , so by a trivial Chernoff bound, we have that $\mathbb{P}_{2,q,\mathcal{I}}[\mathcal{E}_{\text{partsize}}] = O(N^{-\omega(1)})$.

Fixing $s, t \in \{0,1\}^k$ and $b \in \{0,1\}$, if we sample $\mathbf{d} \sim \mathbb{P}_{2,q,\mathcal{I}}$, then we find that

$$\sum_{x \in W_{sb}(\mathbf{d})} \mathbf{d}_t^{(x)} = \sum_{x \in V_s} \mathbf{d}_t^{(x)} \mathbf{1}_{\mathcal{J}_{sb}(\mathbf{d}^{(x)})} =: \sum_{x \in V_s} \mathcal{X}_x,$$

which is the sum of the $|V_s|$ independent random variables $\mathcal{X}_x$. Let $\tilde{\mathcal{X}}_x$ denote $\mathcal{X}_x$ conditioned to lie in the interval $[0.5p|V_t|, 2p|V_t|]$. Since $\mathbb{P}_{2,q,\mathcal{I}}[\mathcal{E}_{\text{degrees}}] = O(N^{-\omega(1)})$, we find that

$$\mathbb{E}[\tilde{\mathcal{X}}_x] = \mathbb{E}[A_{s,t} \mathbf{1}_{\mathcal{J}_{sb}(A_{s,\bullet})} | \mathcal{I}_s(A_{s,\bullet})] + O(N^{-\omega(1)}).$$

By Hoeffding's inequality, we find that

$$\mathbb{P} \left[\left| \left(\sum_{x \in V_s} \tilde{\mathcal{X}}_x \right) - |V_s| \mathbb{E}[A_{s,t} \mathbf{1}_{\mathcal{J}_{sb}(A_{s,\bullet})} | \mathcal{I}_s(A_{s,\bullet})] \right| > \frac{N^2 p}{\sqrt{N}} \log N \right] = O(N^{-\omega(1)}).$$

Thus, since $\mathbb{P}_{2,q,\mathcal{I}}[\mathcal{E}_{\text{degrees}}] = O(N^{-\omega(1)})$, this directly implies that

$$\mathbb{P} \left[\left| \left(\sum_{x \in V_s} \mathcal{X}_x \right) - |V_s| \mathbb{E} [A_{s,t} \mathbf{1}_{\mathcal{J}_{sb}(A_s, \bullet)} | \mathcal{I}_s(A_s, \bullet)] \right| > \frac{N^2 p}{\sqrt{N}} \log N \right] = O(N^{-\omega(1)}),$$

so $\mathbb{P}_{2,q,\mathcal{I}}[\mathcal{E}_{\text{edgecounts}}] = O(N^{-\omega(1)})$.

By [Theorem A.3](#), we have that

$$\mathbb{P}_{2,q,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \right] = \Omega \left((N^2 p)^{-2^{2k-1}} \right).$$

Therefore,

$$\mathbb{P}_{2,m,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} [\mathcal{E}] = \frac{\mathbb{P}_{2,q,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{E} \cap \mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \right]}{\mathbb{P}_{2,q,\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} \left[\mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \right]} = O(N^{-\omega(1)}),$$

so by [Theorem 3.1](#), we have

$$\mathbb{P}_{1,m,\mathcal{I},\kappa_p}[\mathcal{E}] = o(1),$$

where $\mathcal{E} = \mathcal{E}_{\text{degrees}} \vee \mathcal{E}_{\text{partsize}} \vee \mathcal{E}_{\text{edgecount}}$. $\square$

Lemma 4.3. *Adopt all the notation and setup from [Lemma 4.2](#). There exists constants $T' > 1$ and $\delta' > 0$ such that for all $\mathbf{d} \in \mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \setminus (\mathcal{E}_{\text{degrees}} \vee \mathcal{E}_{\text{partsize}} \vee \mathcal{E}_{\text{edgecount}})$, the following is true. Sample $G \sim \mathbb{P}_{\mathbf{d}}$. With probability $1 - o(1)$, the pair*

$$\left(V = \sqcup_{sb \in \{0,1\}^{k+1}} W_{sb}(\mathbf{d}), m \left(\delta_{\{W_{sb}(\mathbf{d})\}_{sb \in \{0,1\}^{k+1}}} (G) \right) \right)$$

is (T', p, δ', τ) -faithful, and $\delta_{\{W_{sb}(\mathbf{d})\}_{sb \in \{0,1\}^{k+1}}} (G) \in \mathcal{D}_{\kappa_p}^{\{W_{sb}(\mathbf{d})\}_{sb \in \{0,1\}^{k+1}}}$.

Proof. Note that $W_{sb}(\mathbf{d})$ doesn't depend on the choice of G , and the faithfulness size condition on it follows trivially from $\mathbf{d} \in \mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \setminus \mathcal{E}_{\text{partsize}}$ and [Theorem B.12](#).

By [Lemma C.7](#), we have that

$$\mathbb{P} \left[\left| m \left(\delta_{\{W_{sb}(\mathbf{d})\}_{sb \in \{0,1\}^{k+1}}} (G) \right)_{sb,tc} - \frac{\left(\sum_{x \in W_{sb}(\mathbf{d})} \mathbf{d}_t^{(x)} \right) \left(\sum_{y \in W_{tc}(\mathbf{d})} \mathbf{d}_s^{(y)} \right)}{m_{s,t}} \right| \geq N^2 p \sqrt{\frac{(pN)^{1/7}}{N}} \log N \right] = o(1),$$

since $\mathbf{d} \in \mathcal{E}_{\text{degrees}}$. Combining this with $\mathbf{d} \in \mathcal{E}_{\text{edgecounts}}$, we have

$$\begin{aligned} \mathbb{P} \left[\left| m \left(\delta_{\{W_{sb}(\mathbf{d})\}_{sb \in \{0,1\}^{k+1}}} (G) \right)_{sb,tc} - \frac{|V_s| |V_t| \mathbb{E} [A_{s,t} \mathbf{1}_{\mathcal{J}_{sb}(A_s, \bullet)} | \mathcal{I}_s(A_s, \bullet)] \mathbb{E} [A_{t,s} \mathbf{1}_{\mathcal{J}_{tc}(A_t, \bullet)} | \mathcal{I}_t(A_t, \bullet)]}{m_{s,t}} \right| \right. \\ \left. \geq N^2 p \sqrt{\frac{(pN)^{1/7}}{N}} \log N \right] = o(1), \end{aligned}$$

which combined with [Theorem B.12](#), shows the faithfulness edgecount condition.

Since $\mathbf{d} \in \mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \setminus \mathcal{E}_{\text{degrees}}$ and the partsizes $|W_{sb}(\mathbf{d})|$ are linear in N by the faithfulness we showed above, [Theorem C.8](#) directly implies that with probability $1 - O(N^{-\omega(1)})$, we have $\delta_{\{W_{sb}(\mathbf{d})\}_{sb \in \{0,1\}^{k+1}}} (G) \in \mathcal{D}_{\kappa_p}^{\{W_{sb}(\mathbf{d})\}_{sb \in \{0,1\}^{k+1}}}$, as desired. $\square$

Lemma 4.4. Fix a real parameter $\theta \in (1/2, 1)$ such that $k = \frac{1}{1-\theta} \in \mathbb{Z}$ and constants $T, \mathcal{T} > 1, \delta > 0$. For all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$ and $\tau \in [\mathcal{T}^{-1}, \mathcal{T}]$, the following is true.

Consider a (T, p, δ, τ) -faithful pair $(V = \sqcup_{s \in \{0,1\}^k} V_s, m)$ with $|V| = N$. Then, m is solvable with respect to the partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$, and there exists a solution $q \in (0, 1)^{\{0,1\}^k \times \{0,1\}^k}$ such that the following is true. Let

$$\mathcal{E}_{\text{partsize}} := \left\{ \mathbf{d} \in \mathcal{D}_{\mathcal{I}}^{\{V_s\}_{s \in \{0,1\}^k}} : ||W_{sb}(\mathbf{d})| - |V_s| \mathbb{P}[\mathcal{J}_{sb}(A_{s,\bullet})|\mathcal{I}_s(A_{s,\bullet})]| > \sqrt{N} \log N \text{ for some } sb \in \{0,1\}^{k+1} \right\}$$

where we sample $A_{s,t} \sim \text{Bin}(|V_t| - \mathbf{1}_{s=t}, q_{s,t})$ all independently. Then,

$$\mathbb{P}_{1,m,\mathcal{I},\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathcal{E}_{\text{partsize}}] = o(1).$$

Proof. Note that [Theorem B.14](#) implies that there is some constant $T' > 1$ such that $V = \sqcup_{s \in \{0,1\}^k} V_s$ is (T', p) -tolerant, and $m \in \mathcal{M}_{T',p}^{\{V_s\}_{s \in \{0,1\}^k}}$ is solvable with respect to the partitioned set $V = \sqcup_{s \in \{0,1\}^k} V_s$. Let $q \in (0, 1)^{\{0,1\}^k \times \{0,1\}^k}$ be the solution specified by [Theorem B.14](#).

If we sample $\mathbf{d} \sim \mathbb{P}_{2,q,\mathcal{I}}$, we see that

$$|W_{sb}(\mathbf{d})| \sim \text{Bin}(|V_s|, \mathbb{P}[\mathcal{J}_{sb}(A_{s,\bullet})|\mathcal{I}_s(A_{s,\bullet})])$$

where we sample $A_{s,t} \sim \text{Bin}(|V_t| - \mathbf{1}_{s=t}, q_{s,t})$ independently. By [Theorem B.14](#), there exists a constant $\phi \in (0, 1/2)$ such that

$$\mathbb{P}[\mathcal{J}_{sb}(A_{s,\bullet})|\mathcal{I}_s(A_{s,\bullet})] \in (\phi, 1 - \phi)$$

for all N , so by a trivial Chernoff bound, we have that $\mathbb{P}_{2,q,\mathcal{I}}[\mathcal{E}_{\text{partsize}}] = O(N^{-\omega(1)})$.

The finish is identical to that of [Lemma 4.2](#), so we omit it for brevity. This completes the proof. $\square$

Lemma 4.5. Adopt all the notation and setup from [Lemma 4.4](#). For all $\mathbf{d} \in \mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \setminus \mathcal{E}_{\text{partsize}}$, we have

$$|W_{sb}(\mathbf{d})| \geq_{bb} N \left(\nu_{sb} + (-1)^b \zeta_{k,\mathcal{T}} \right)$$

for all $sb \in \{0,1\}^{k+1}$.

Proof. This is a trivial corollary of [Theorem B.14](#). $\square$

Proof of Theorem 4.1. We prove the modified result where we have potentially different values of T', δ for each k . The stated result can be recovered by taking the maximum over all T' values, and minimum over all δ values.

We proceed by induction on k , and begin by proving the base case $k = 1$. The part size condition of faithfulness is true by definition (say with $T' = 1$ and $\delta = 1/2$). Note that

$$m_{s,t}(\delta_1(G)) \sim (1 + \mathbf{1}_{s=t}) \cdot \text{Bin}\left(\frac{|V_s|(|V_t| - \mathbf{1}_{s=t})}{1 + \mathbf{1}_{s=t}}, p\right)$$

for $s, t \in \{0,1\}$, so by a trivial Chernoff bound, with probability $1 - o(1)$, we have

$$|m_{s,t} - p|V_s||V_t|| \leq \sqrt{N^2 p \log N},$$

so the faithfulness condition on $m(\delta_1(G))$ is true with $\delta = 0.49$ and $T' = 1$. Thus, $(V = \sqcup_{s \in \{0,1\}} V_s(G), m(\delta_1(G)))$ is $(1, p, 0.49, \tau)$ -faithful. Finally, note that for $s, t \in \{0,1\}$ and $x \in V_s$, $\delta_1(G)_t^{(x)} \sim \text{Bin}(|V_t| - \mathbf{1}_{s=t}, p)$, so the condition on $\delta_1(G)$ follows trivially by a Chernoff bound.

Now, suppose the result is true for some $k \in \left[1, \frac{1}{1-\theta}\right)$ with constants $T' > 1$ and $\delta > 0$. We will show the result for $k+1$, with constants T'', δ' given by [Lemma 4.3](#), where we plug in $\max\{T', T\}$ for the T -value in the setup, and δ for the δ -value in the setup.

Indeed, let $\mathcal{F}$ denote the event containing politicized graphs G with vertex set V and $V_0(G) = V_0$ and $V_1(G) = V_1$ such that the pair $\left(V = \sqcup_{s \in \{0,1\}^{k+1}} V_s(G), m(\delta_{k+1}(G))\right)$ is (T'', p, δ', τ) -faithful, and $\delta_{k+1}(G) \in \mathcal{D}_{\kappa_p}^{\{V_s(G)\}_{s \in \{0,1\}^{k+1}}}$. Concretely, we will show that $\mathbb{P}_{G \sim \mathbb{G}(V,p)}[G \in \mathcal{F}] = 1 - o(1)$, which would complete the inductive step.

For any (T', p, δ, τ) -faithful pair $\Pi := \left(V = \sqcup_{s \in \{0,1\}^k} V_s, m\right)$, let $\mathcal{E}_\Pi$ denote the associated event $\mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \setminus (\mathcal{E}_{\text{degrees}} \vee \mathcal{E}_{\text{partsize}} \vee \mathcal{E}_{\text{edgecount}})$ from the setup of [Lemma 4.2](#), and let $\mathcal{E}$ be the union of all of these over all such pairs Π . We see that

$$\begin{aligned} \mathbb{P}_{G \sim \mathbb{G}(V,p)}[\delta_k(G) \in \mathcal{E}] &\geq \\ \sum_{\substack{\Pi = (V = \sqcup_{s \in \{0,1\}^k} V_s, m) \\ \Pi \text{ is } (T', p, \delta, \tau)\text{-faithful}}} \mathbb{P}[V_s(G) = V_s \wedge m(\delta_k(G)) = m \wedge \kappa_p(\delta_k(G))] &\mathbb{P}_{1,m,\mathcal{I},\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathcal{E}_\Pi]. \end{aligned}$$

By the inductive hypothesis, we have that

$$\sum_{\substack{\Pi = (V = \sqcup_{s \in \{0,1\}^k} V_s, m) \\ \Pi \text{ is } (T', p, \delta, \tau)\text{-faithful}}} \mathbb{P}[V_s(G) = V_s \wedge m(\delta_k(G)) = m \wedge \kappa_p(\delta_k(G))] = 1 - o(1),$$

so combining with [Lemma 4.2](#), we have $\mathbb{P}_{G \sim \mathbb{G}(V,p)}[\delta_k(G) \in \mathcal{E}] = 1 - o(1)$. Now, we see that

$$\mathbb{P}_{G \sim \mathbb{G}(V,p)}[G \in \mathcal{F}] \geq \sum_{\mathbf{d} \in \mathcal{E}} \mathbb{P}_{G \sim \mathbb{G}(V,p)}[\delta_k(G) = \mathbf{d}] \mathbb{P}_{G \sim \mathbb{P}_{\mathbf{d}}}[G \in \mathcal{F}] = 1 - o(1),$$

based on our previous result $\mathbb{P}_{G \sim \mathbb{G}(V,p)}[\delta_k(G) \in \mathcal{E}] = 1 - o(1)$ and [Lemma 4.3](#). This completes the inductive step in the case $k \in \left[1, \frac{1}{1-\theta}\right)$.

Now, suppose the result is true for $k = \frac{1}{1-\theta}$ with constants $T' > 1$ and $\delta > 0$. Let $\mathcal{H}$ denote the event containing politicized graphs G with vertex set V and $V_0(G) = V_0$ and $V_1(G) = V_1$ such that $|V_s| \geq \overline{s_k s_k} N(\nu_s + (-1)^{s_k} \zeta_{k-1, \tau})$ for all $s \in \{0,1\}^{k+1}$. We will show that $\mathbb{P}_{G \sim \mathbb{G}(V,p)}[G \in \mathcal{H}] = 1 - o(1)$, which would complete the inductive step.

For any (T', p, δ, τ) -faithful pair $\Pi := \left(V = \sqcup_{s \in \{0,1\}^k} V_s, m\right)$, let $\mathcal{E}_\Pi$ denote the associated event $\mathcal{D}_m^{\{V_s\}_{s \in \{0,1\}^k}} \setminus \mathcal{E}_{\text{partsize}}$ from the setup of [Lemma 4.4](#), and let $\mathcal{E}$ be the union of all of these over all such pairs Π . We see that

$$\begin{aligned} \mathbb{P}_{G \sim \mathbb{G}(V,p)}[\delta_k(G) \in \mathcal{E}] &\geq \\ \sum_{\substack{\Pi = (V = \sqcup_{s \in \{0,1\}^k} V_s, m) \\ \Pi \text{ is } (T', p, \delta, \tau)\text{-faithful}}} \mathbb{P}[V_s(G) = V_s \wedge m(\delta_k(G)) = m \wedge \kappa_p(\delta_k(G))] &\mathbb{P}_{1,m,\mathcal{I},\kappa_p}^{\{V_s\}_{s \in \{0,1\}^k}}[\mathcal{E}_\Pi]. \end{aligned}$$

By the inductive hypothesis, we have that

$$\sum_{\substack{\Pi = (V = \sqcup_{s \in \{0,1\}^k} V_s, m) \\ \Pi \text{ is } (T', p, \delta, \tau)\text{-faithful}}} \mathbb{P}[V_s(G) = V_s \wedge m(\delta_k(G)) = m \wedge \kappa_p(\delta_k(G))] = 1 - o(1),$$

so combining with [Lemma 4.4](#), we have $\mathbb{P}_{G \sim \mathbb{G}(V,p)}[\delta_k(G) \in \mathcal{E}] = 1 - o(1)$. Now, we see that

$$\mathbb{P}_{G \sim \mathbb{G}(V,p)}[G \in \mathcal{H}] \geq \sum_{\mathbf{d} \in \mathcal{E}} \mathbb{P}_{G \sim \mathbb{G}(V,p)}[\delta_k(G) = \mathbf{d}] \mathbb{P}_{G \sim \mathbb{P}_{\mathbf{d}}}[G \in \mathcal{H}] = \sum_{\mathbf{d} \in \mathcal{E}} \mathbb{P}_{G \sim \mathbb{G}(V,p)}[\delta_k(G) = \mathbf{d}] = 1 - o(1),$$

based on our previous result $\mathbb{P}_{G \sim \mathbb{G}(V,p)}[\delta_k(G) \in \mathcal{E}] = 1 - o(1)$ and [Lemma 4.5](#). This completes the inductive step in the case $k = \frac{1}{1-\theta}$.

This completes the proof. $\square$

We now package [Theorem 4.1](#) into a result that allows us to bootstrap using the techniques of [\[5\]](#).

Theorem 4.6. *Fix a real parameter $\theta \in (1/2, 1)$ and constants $T, \mathcal{T} > 1$. For all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$ and $\tau \in [\mathcal{T}^{-1}, \mathcal{T}]$, the following is true.*

Consider a partitioned set $V = V_0 \sqcup V_1$ with $|V| = N$ and $|V_0| = \lfloor N/2 \rfloor + \lfloor \tau\sqrt{N} \rfloor$. Sample a graph $G \sim \mathbb{G}(V, p)$, and politicize it according to $V_0(G) = V_0$ and $V_1(G) = V_1$. Letting $k = \lfloor \frac{1}{1-\theta} \rfloor + 1$, we have

$$\sum_{s \in \{0,1\}^k} (-1)^{s_{k-1}} |V_s(G)| > \frac{N}{\sqrt{pN}} \log N$$

with probability $1 - o(1)$.

Proof. Choose $T' > 1$ and $\delta > 0$ as per [Theorem 4.1](#). First consider the case where $\frac{1}{1-\theta} \notin \mathbb{Z}$. In that case, [Theorem 4.1](#) tells us that

$$\left| |V_s(G)| - \tilde{n}_s(N) - \tau\sqrt{N}\sqrt{pN}^{k-1}\varepsilon_s \right| \leq T\sqrt{N}\sqrt{pN}^{k-1}N^{-\delta}$$

for all $s \in \{0,1\}^k$, with probability $1 - o(1)$. Therefore, by the bit-flip symmetry of $\tilde{n}_s(N)$ given in [Theorem B.11](#) and [Lemma B.10](#), we see that

$$\begin{aligned} \sum_{s \in \{0,1\}^k} (-1)^{s_{k-1}} |V_s(G)| &= \tau\sqrt{N}\sqrt{pN}^{k-1} \sum_{s \in \{0,1\}^k} (-1)^{s_{k-1}} \varepsilon_s + o\left(\sqrt{N}\sqrt{pN}^{k-1}\right) \\ &= \Omega\left(\sqrt{N}\sqrt{pN}^{k-1}\right) \\ &= \Omega\left(\frac{N}{\sqrt{pN}} \log N\right), \end{aligned}$$

which completes the proof in this case.

Now consider the case where $\frac{1}{1-\theta} \in \mathbb{Z}$. In that case, [Theorem 4.1](#) tells us that

$$|V_s| \geq_{\overline{s_{k-1}} s_{k-1}} N(\nu_s + (-1)^{s_{k-1}} \zeta_{k-1, \tau})$$

with probability $1 - o(1)$. Therefore by the bit-flip symmetry present in $\nu_\bullet$, we have

$$\sum_{s \in \{0,1\}^k} (-1)^{s_{k-1}} |V_s(G)| > 2^k N \zeta_{k-1, \tau} > \frac{N}{\sqrt{pN}} \log N,$$

as desired. $\square$

5. FINISHING

We have the following result that shows that the process ends with all vertices the same color in $2\lfloor \frac{1}{1-\theta} \rfloor + 2$ steps.

Theorem 5.1. Fix a real parameter $\theta \in (1/2, 1)$ and constants $T, \mathcal{T} > 1$. For all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$ and $\tau \in [\mathcal{T}^{-1}, \mathcal{T}]$, the following is true.

Consider a partitioned set $V = V_0 \sqcup V_1$ with $|V| = N$ and $|V_0| = \lfloor N/2 \rfloor + \lfloor \tau\sqrt{N} \rfloor$. Sample a graph $G \sim \mathbb{G}(V, p)$, and politicize it according to $V_0(G) = V_0$ and $V_1(G) = V_1$. Letting $k = 2\lfloor \frac{1}{1-\theta} \rfloor + 3$, we have $V_s(G) = \emptyset$ for all $s \in \{0, 1\}^k$ with $s_{k-1} = 1$ with probability $1 - o(1)$.

To prove this, we borrow the results of section 4 of [5]. We say a graph G is (p, β) -jumbled if for any vertex subsets $U, V \subset V(G)$, we have

$$|e(U, V) - p|U||V|| \leq \beta\sqrt{|U||V|}.$$

The following result is a direct corollary of Lemmas 2.2 and 4.1 of [5].

Lemma 5.2 ([5], Lemma 2.2 and Lemma 4.1). Fix a real parameter $\theta \in (1/2, 1)$ and a constant $T > 1$. For all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$, the following is true.

Fix a vertex set V with $|V| = N$ and sample $G \sim \mathbb{G}(V, p)$. With probability $1 - o(1)$, the minimum degree of G is at least $0.9np$, and G is $(p, \sqrt{Np \log N})$ -jumbled.

We are now ready to prove the main result.

Proof of Theorem 5.1. For any positive integer k' , define

$$\Delta_{k'} := \sum_{s \in \{0, 1\}^{k'}} (-1)^{s_{k'-1}} |V_s(G)|.$$

By Theorem 4.6, Lemma 5.2, and Lemma 4.2(i) of [5] with $\delta = 0.1$ and $\beta = \sqrt{pN \log N}$, we have with probability $1 - o(1)$ that

$$\Delta_{\lfloor \frac{1}{1-\theta} \rfloor + 2} \geq 0.95N.$$

An easy induction with Lemma 4.2(ii) of [5] tells us that

$$\Delta_{\lfloor \frac{1}{1-\theta} \rfloor + 2 + \ell} \geq \left(1 - 0.05 \left(\frac{1600 \log N}{Np}\right)^\ell\right) N$$

for all nonnegative integers ℓ . Thus, for $\ell > \frac{1}{1-\theta}$, we have

$$\Delta_{\lfloor \frac{1}{1-\theta} \rfloor + 2 + \ell} \geq N - 0.5.$$

Thus,

$$\Delta_{2\lfloor \frac{1}{1-\theta} \rfloor + 3} = N,$$

which completes the proof. $\square$

APPENDIX A. RESULTS ON BINOMIAL RANDOM VARIABLES

We have the following two-tiered estimate for the PDF of a binomial random variable.

Lemma A.1. Fix a real parameter $\theta \in (1/2, 1)$ and a real constant $T > 1$. For all sufficiently large $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^{-\theta})$, the following is true.

Select $n' \in \mathbb{Z}$ and $p' \in (0, 1)$ such that $|n' - n| < T\frac{n}{\sqrt{pn}}$ and $|p' - p| < T\frac{p}{\sqrt{pn}}$. There exists real numbers $\mathcal{N}_1$ and $\mathcal{N}_2$ such that uniformly for $x \in [-\sqrt{pn} \log n, \sqrt{pn} \log n] \cap (\mathbb{Z} - pn)$, we have

$$\mathbb{P}[\text{Bin}(n', p') = pn + x] = (1 + O(p \log^2 n)) \mathcal{N}_1 \left(\frac{p'(n' - pn)}{(1 - p')pn} \right)^x \frac{(pn)!(pn)^x}{(pn + x)!}$$

and

$$\mathbb{P}[\text{Bin}(n', p') = pn + x] = (1 + O(\log^3 n / \sqrt{pn})) \mathcal{N}_2 e^{-(x - \alpha\sqrt{pn})^2 / 2pn}$$

where $\alpha = \sqrt{pn} \log \left(\frac{p'(n' - pn)}{(1 - p')pn} \right)$.

Proof. Uniformly for $x \in [-\sqrt{pn} \log n, \sqrt{pn} \log n]$, we have

$$\begin{aligned}
\mathbb{P}[\text{Bin}(n', p') = pn + x] &= \frac{n'!}{(pn+x)!(n'-pn-x)!} p'^{pn+x} (1-p')^{n'-pn-x} \\
&= \frac{n'!}{(pn)!(n'-pn)!} p'^{pn} (1-p')^{n'-pn} \frac{(n'-pn)!}{(n'-pn-x)!} \frac{(pn)!}{(pn+x)!} \left(\frac{p'}{1-p'}\right)^x \\
&= (1 + O(p \log^2 n)) \frac{n'!}{(pn)!(n'-pn)!} p'^{pn} (1-p')^{n'-pn} (n'-pn)^x \frac{(pn)!}{(pn+x)!} \left(\frac{p'}{1-p'}\right)^x \\
&= (1 + O(p \log^2 n)) \mathcal{N}_1 \left(\frac{p'(n'-pn)}{(1-p')pn} \right)^x \frac{(pn)!(pn)^x}{(pn+x)!},
\end{aligned}$$

for some normalization $\mathcal{N}_1$ independent of x . This proves the first equality.

For the second equality, we use the fact that

$$n! = \sqrt{2\pi n} (n/e)^n (1 + O(1/n))$$

as $n \rightarrow \infty$. Indeed, uniformly for $x \in [-\sqrt{pn} \log n, \sqrt{pn} \log n] \cap (\mathbb{Z} - pn)$, we have

$$\begin{aligned}
\frac{(pn+x)!}{(pn)!(pn)^x} &= (1 + O(1/pn)) e^{-x} \left(1 + \frac{x}{pn}\right)^{pn+x+\frac{1}{2}} \\
&= (1 + O(1/pn)) e^{-x} \exp\left(\left(pn+x+\frac{1}{2}\right) \left[\frac{x}{pn} - \frac{x^2}{2(pn)^2} + O\left(\frac{\log^3 n}{(pn)^{3/2}}\right)\right]\right) \\
&= \left(1 + O\left(\frac{\log^3 n}{\sqrt{pn}}\right)\right) \exp\left(\frac{x^2}{2pn}\right),
\end{aligned}$$

which combined with the previous equality implies the second equality. $\square$

Theorem A.2. Fix a real parameter $\theta \in (1/2, 1)$, a positive integer $d \geq 1$, an integer matrix $M \in \mathbb{Z}^{r \times d}$ with pairwise orthogonal rows, an r -dimensional vector inequality relation $\geq' \in \{\geq, >\}^r$, a monomial $\rho \in \mathbb{R}[x_1, \dots, x_d]$, and a real constant $T > 1$. For all sufficiently large $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^\theta)$, the following is true.

Select $\eta \in ((T^{-1}n, Tn) \cap \mathbb{Z})^d$ such that $\|M\eta\|_\infty < T \frac{n}{\sqrt{pn}}$, and select $\alpha \in (-T, T)^d$. Then,

$$\mathbb{E}[\rho(A - p\eta) \mathbf{1}_{MA \geq' 0}] = \mathbb{E}[\rho(X) \mathbf{1}_{MX \geq -pM\eta}] + O(\sqrt{pn}^{\deg \rho - 1} \log^{1+\deg \rho + d} n)$$

where $A_t \sim \text{Bin}\left(\eta_t, \frac{\frac{p}{1-p} e^{\alpha_t / \sqrt{p\eta_t}}}{1 + \frac{p}{1-p} e^{\alpha_t / \sqrt{p\eta_t}}}\right)$ are all independently sampled, and $X_t \sim \mathcal{N}(\sqrt{p\eta_t} \alpha_t, p\eta_t)$ are all independently sampled.

Proof. Call a convex region $\mathcal{C}$ region (each boundary can be either open or closed) in $\mathbb{R}^d$ good if it is bounded by pairwise orthogonal planes with integer normal vectors whose entries are bounded by $\|M\|_\infty$, and distance to the origin bounded by $\sqrt{pn \log n}$. For such a region, define

$$\mathcal{P}_1(\mathcal{C}) := \bigcup_{x \in \prod_{t \in [d]} (\mathbb{Z} - p\eta_t) \cap [-\sqrt{p\eta_t} \log \eta_t, \sqrt{p\eta_t} \log \eta_t]} \prod_{x \in \mathcal{C}} [x_t, x_t + 1),$$

and

$$\mathcal{P}_2(\mathcal{C}) := \left\{ x \in \prod_{t \in [d]} [-\sqrt{p\eta_t} \log \eta_t, \sqrt{p\eta_t} \log \eta_t] : x \in \mathcal{C} \right\},$$

and

$$\mathcal{P}_3(\mathcal{C}) := \left\{ x \in \mathbb{R}^d : x \in \mathcal{C} \right\}.$$

Note that $\mathbb{P}[|A_t - p\eta_t| > \sqrt{p\eta_t} \log \eta_t] = O(n^{-\omega(1)})$ by a standard Chernoff bound, so by [Lemma A.1](#), we have

$$\mathbb{E}[\rho(A - p\eta)\mathbf{1}_{MA \geq 0}] = \frac{\sum_{\substack{x \in \prod_{t \in [d]} (\mathbb{Z} - p\eta_t) \cap [-\sqrt{p\eta_t} \log \eta_t, \sqrt{p\eta_t} \log \eta_t] \\ Mx \geq' -pM\eta}} \rho(x)f(x)}{\sum_{x \in \prod_{t \in [d]} (\mathbb{Z} - p\eta_t) \cap [-\sqrt{p\eta_t} \log \eta_t, \sqrt{p\eta_t} \log \eta_t]} f(x)} + O(\sqrt{pn}^{\deg \rho - 1} \log^{3+\deg \rho} n),$$

where

$$f(x) := \exp \left(- \sum_{t \in [d]} \frac{(x_t - \alpha_t \sqrt{p\eta_t})^2}{2p\eta_t} \right).$$

The first step is to turn the summations into integrals over $\mathcal{P}_1$.

Note that for $x \in \prod_{t \in [d]} [-\sqrt{p\eta_t} \log \eta_t, \sqrt{p\eta_t} \log \eta_t]$, we have

$$f(x + \varepsilon) = f(x) + O(\log n / \sqrt{pn})$$

for any $\varepsilon \in [0, 1]^d$. In particular, this implies that

$$f(x) = \int_{[0,1]^d} d\varepsilon f(x + \varepsilon) + O(\log n / \sqrt{pn})$$

and

$$\rho(x)f(x) = \int_{[0,1]^d} d\varepsilon \rho(x + \varepsilon)f(x + \varepsilon) + O(\sqrt{pn}^{\deg \rho - 1} \log^{1+\deg \rho} n)$$

for all $x \in \prod_{t \in [d]} [-\sqrt{p\eta_t} \log \eta_t, \sqrt{p\eta_t} \log \eta_t]$. Thus, letting $\mathcal{C}_M = \{x \in \mathbb{R}^d : Mx \geq' -pM\eta\}$, we have

$$\mathbb{E}[\rho(A - p\eta)\mathbf{1}_{MA \geq 0}] = \frac{\int_{\mathcal{P}_1(\mathcal{C}_M)} dx \rho(x)f(x) + O(\sqrt{pn}^{\deg \rho + d - 1} \log^{1+\deg \rho + d} n)}{\int_{\mathcal{P}_1(\mathbb{R}^d)} dx f(x) + O(\sqrt{pn}^{d-1} \log^{1+d} n)} + O(\sqrt{pn}^{\deg \rho - 1} \log^{3+\deg \rho} n).$$

There is a constant C such that for any good $\mathcal{C}$, $\mathcal{P}_1 \setminus \mathcal{P}_2$ and $\mathcal{P}_2 \setminus \mathcal{P}_1$ are both contained in $B(\partial \mathcal{P}_2, C)$, and $\text{vol } B(\partial \mathcal{P}_2, C) = O(\sqrt{pn}^{d-1} \log^{d-1} n)$. Therefore,

$$\mathbb{E}[\rho(A - p\eta)\mathbf{1}_{MA \geq 0}] = \frac{\int_{\mathcal{P}_2(\mathcal{C}_M)} dx \rho(x)f(x) + O(\sqrt{pn}^{\deg \rho + d - 1} \log^{1+\deg \rho + d} n)}{\int_{\mathcal{P}_2(\mathbb{R}^d)} dx f(x) + O(\sqrt{pn}^{d-1} \log^{1+d} n)} + O(\sqrt{pn}^{\deg \rho - 1} \log^{3+\deg \rho} n).$$

Note that $\mathcal{P}_2 \subset \mathcal{P}_3$ and $\mathcal{P}_3 \setminus \mathcal{P}_2 \subset \{x \in \mathbb{R}^d : \|x\|_\infty > C\sqrt{pn} \log n\}$ for some constant C , and

$$\begin{aligned} \int_{\|x\|_\infty > C\sqrt{pn} \log n} dx f(x) &= O(n^{-\omega(1)}) \\ \int_{\|x\|_\infty > C\sqrt{pn} \log n} dx \|\rho(x)\|_\infty f(x) &= O(n^{-\omega(1)}). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}[\rho(A - p\eta)\mathbf{1}_{MA \geq 0}] &= \frac{\int_{\mathcal{P}_3(\mathcal{C}_M)} dx \rho(x)f(x) + O(\sqrt{pn}^{\deg \rho + d - 1} \log^{1+\deg \rho + d} n)}{\int_{\mathcal{P}_3(\mathbb{R}^d)} dx f(x) + O(\sqrt{pn}^{d-1} \log^{1+d} n)} + O(\sqrt{pn}^{\deg \rho - 1} \log^{3+\deg \rho} n) \\ &= \frac{\int_{x \in \mathcal{C}_M} dx \rho(x)f(x)}{\int_{x \in \mathbb{R}^d} dx f(x)} + O(\sqrt{pn}^{\deg \rho - 1} \log^{1+\deg \rho + d} n), \end{aligned}$$

as desired. $\square$

We have the following result that allows us to lower bound edge count probabilities.

Theorem A.3. Fix a real parameter $\theta \in (1/2, 1)$, a positive integer $d \geq 1$, an integer matrix $M \in \mathbb{Z}^{r \times d}$ with pairwise orthogonal rows, an r -dimensional vector inequality relation $\geq' \in \{\geq, >\}$, and a real constant $T > 1$. For all sufficiently large $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^{-\theta})$, the following is true.

Select $\eta \in ((T^{-1}n, Tn) \cap \mathbb{Z})^d$ such that $\|M\eta\|_\infty < T\frac{n}{\sqrt{pn}}$ and select $m \in (T^{-1}n, Tn)$. Select $q \in \left(p - T\frac{p}{\sqrt{pn}}, p + T\frac{p}{\sqrt{pn}}\right)^d$ such that $m\mathbb{E}[A|MA \geq' 0] \in \mathbb{Z}^d$ where we sample $A_t \sim \text{Bin}(\eta_t, q_t)$ all independently.

Let ρ be the probability distribution on $\mathbb{Z}^d$ given by sampling $A_t \sim \text{Bin}(\eta_t, q_t)$ all independently, and then conditioning on $MA \geq' 0$.

Sample m independent copies $A^{(1)}, \dots, A^{(m)} \sim \rho$, and let $S = A^{(1)} + \dots + A^{(m)}$. Then,

$$\mathbb{P}[S = m\mathbb{E}_{A \sim \rho}[A]] = \Omega((n^2p)^{-d/2}).$$

Proof. Let B be the random variable distributed as $A - \mathbb{E}_{A \sim \rho}[A]$ where A is sampled from ρ . Sample m independent copies $B^{(1)}, \dots, B^{(m)}$ and let

$$S' = B^{(1)} + \dots + B^{(m)}.$$

It suffices to lower bound the probability $\mathbb{P}[S' = 0]$. Note that S' is supported on a subset of $\mathbb{Z}^d$, so we have

$$\mathbb{P}[S' = 0] = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} dt \mathbb{E}[e^{it \cdot S'}].$$

We now estimate the characteristic function $\mathbb{E}[e^{it \cdot S'}]$ for various regimes of t .

To do this, we first need to prove regularity conditions on the covariance matrix $\text{Cov}B$. Indeed, let $\alpha_t = \sqrt{p\eta_t} \log\left(\frac{q_t}{1-q_t} \frac{1-p}{p}\right)$ and sampling independent $X_t \sim \mathcal{N}(\sqrt{p\eta_t}\alpha_t, p\eta_t)$ for all $t \in [d]$, we see by [Theorem A.2](#) that

$$\text{Cov}B = \text{Cov}[X|MX \geq -pM\eta] + O(\sqrt{pn} \log^{3+d} n).$$

It is easy to see that $\text{Cov}[X|MX \geq -pM\eta]$ is positive definite, as if it wasn't, some linear combination of the X_t s would be identically zero post conditioning on $MX \geq -pM\eta$, which can't happen since the inequalities are pairwise orthogonal and the support of the conditioned X therefore has full dimension. Note that $\frac{1}{pn} \text{Cov}[X|MX \geq -pM\eta]$ is a continuous function of $(\eta_t/n)_{t \in [d]}$ and $(\alpha_t)_{t \in [d]}$ that is independent of choice of n or p . Thus, since the domains of $(\eta_t/n)_{t \in [d]}$ and $(\alpha_t)_{t \in [d]}$ are supported in a region bounded by constants independent of n and p , we see that there is some constant C such that the entries of $\frac{1}{pn} \text{Cov}[X|MX \geq -pM\eta]$ and its inverse are bounded by C in absolute value. This implies that the entries of $\frac{1}{pn} \text{Cov}B$ and its inverse are bounded by $2C$ in absolute value.

Using this, the multidimensional Berry-Essen theorem tells us that for all convex sets $U \subset \mathbb{R}^d$, we have

$$|\mathbb{P}[S \in U] - \mathbb{P}[Z \in U]| \leq \frac{C_1}{n^{1/2-0.01/d}}$$

for some other constant C_1 , where $Z \sim \mathcal{N}(0, m\text{Cov}B)$. Fix any $t \in [-\pi, \pi]^d$ with $\|t\|_\infty \leq \frac{n^{0.01/d}}{\sqrt{n^2p}}$.

For all $\ell \in \mathbb{Z}$, let

$$U_\ell = \{x \in \mathbb{R}^d : 2\pi\ell/\lfloor n^{1/6} \rfloor \leq t \cdot x < 2\pi(\ell+1)/\lfloor n^{1/6} \rfloor\},$$

which is a convex set. Furthermore, we have $t \cdot S' < \|t\|_\infty \sqrt{n^2pn}^{0.01/d} < n^{0.02/d}$ with probability $1 - O(n^{-\omega(1)})$. Thus, we have

$$\mathbb{E}[e^{it \cdot S'}] = O(n^{-\omega(1)}) + \sum_{\substack{\ell \in \mathbb{Z} \\ |\ell| < n^{0.02/d}}} (e^{2\pi i \ell / k} + O(1/k)) \mathbb{P}[S \in U_\ell]$$

$$\begin{aligned}
&= O(n^{-1/6}) + \sum_{\substack{\ell \in \mathbb{Z} \\ |\ell| < n^{0.02/d}}} e^{2\pi i \ell / k} \left(\mathbb{P}[Z \in U_\ell] + O(n^{-1/2+0.01/d}) \right) \\
&= O(n^{-1/6}) + \mathbb{E}[e^{it \cdot Z}],
\end{aligned}$$

so

$$\mathbb{P}[S' = 0] = O(n^{-0.1}(n^2 p)^{-d/2}) + \frac{1}{(2\pi)^d} \int_{\substack{t \in [-\pi, \pi]^d \\ \|t\|_\infty \leq n^{0.01/d}/\sqrt{n^2 p}}} dt \mathbb{E}[e^{it \cdot Z}] + \frac{1}{(2\pi)^d} \int_{\substack{t \in [-\pi, \pi]^d \\ \|t\|_\infty > n^{0.01/d}/\sqrt{n^2 p}}} dt \mathbb{E}[e^{it \cdot S'}].$$

There is a constant C_2 such that

$$|\mathbb{E}[e^{it \cdot Z}]| \leq \exp(-C_2 n^2 p \|t\|_\infty^2),$$

which implies that

$$\int_{\substack{t \in \mathbb{R}^d \\ \|t\|_\infty > n^{0.01/d}/\sqrt{n^2 p}}} dt \mathbb{E}[e^{it \cdot Z}] = O(n^{-\omega(1)}).$$

Thus,

$$\begin{aligned}
\mathbb{P}[S' = 0] &= \frac{1}{(2\pi)^d} \int_{t \in \mathbb{R}^d} dt \mathbb{E}[e^{it \cdot Z}] + \frac{1}{(2\pi)^d} \int_{\substack{t \in [-\pi, \pi]^d \\ \|t\|_\infty > n^{0.01/d}/\sqrt{n^2 p}}} dt \mathbb{E}[e^{it \cdot S'}] + o((n^2 p)^{-d/2}) \\
&= \mathbb{P}[Z = 0] + \frac{1}{(2\pi)^d} \int_{\substack{t \in [-\pi, \pi]^d \\ \|t\|_\infty > n^{0.01/d}/\sqrt{n^2 p}}} dt \mathbb{E}[e^{it \cdot S'}] + o((n^2 p)^{-d/2}) \\
&= \Omega((n^2 p)^{-d/2}) + \frac{1}{(2\pi)^d} \int_{\substack{t \in [-\pi, \pi]^d \\ \|t\|_\infty > n^{0.01/d}/\sqrt{n^2 p}}} dt \mathbb{E}[e^{it \cdot S'}].
\end{aligned}$$

In order to estimate this error term, we use the following trick.

Let $\mu_1, \dots, \mu_r \in \mathbb{Z}^d$ be the rows of M . Denote $y \in \prod_{t \in [d]} (\mathbb{Z} - p\eta_t)$ as *good* if

$$\frac{y}{\sqrt{pn}} \in [T+2]\mu_1 + \dots + [T+2]\mu_r + \left[0, \frac{1}{d\|M\|_\infty}\right]^d.$$

It is easy to see that $M(p\eta + y) \geq' 0$ for all good y .

Let $f(x)$ be defined as in the proof of [Theorem A.2](#). Following the proof of [Theorem A.2](#), uniformly for good y , we have

$$\mathbb{P}[A = p\eta + y] = (1 + O(\log^d n / \sqrt{pn})) \frac{f(y)}{\int_{\substack{x \in \mathbb{R}^d \\ Mx \geq' -pM\eta}} dx f(x)} = \Omega((pn)^{-d/2}).$$

Armed with this result, we are now ready to provide the relevant estimate of $|\mathbb{E}[e^{it \cdot S'}]| = |\mathbb{E}[e^{it \cdot (S - mp\eta)}]| = |\mathbb{E}[e^{it \cdot (A - p\eta)}]|^m$. The above analysis implies that we can decompose $A - p\eta$ into

$$A - p\eta = \mathbf{1}_{\mathcal{E}} T_1 + \mathbf{1}_{\mathcal{E}^c} T_2$$

where $\mathcal{E}$ is an event of probability $C_3 \in (0, 1)$, where C_3 is a constant independent of n, p , and T_1 is given by the uniform distribution on all good elements, and $T_1, \mathbf{1}_{\mathcal{E}}$ and $T_2, \mathbf{1}_{\mathcal{E}^c}$ are pairs of independent random variables. Thus, we have

$$|\mathbb{E}[e^{it \cdot (A - p\eta)}]| = |C_3 \mathbb{E}[e^{it \cdot T_1}] + (1 - C_3) \mathbb{E}[e^{it \cdot T_2}]| \leq 1 - C_3 + C_3 |\mathbb{E}[e^{it \cdot T_1}]|.$$

Note that the set of good elements forms a discrete rectangular prism. Let its width in the j th dimension be $W_j\sqrt{pn} \in \mathbb{Z}$ (note that $W_j = \Theta(1)$). Then, we see that

$$|\mathbb{E}[e^{it \cdot T_1}]| = \prod_{j=1}^d \frac{1}{W_j\sqrt{pn}} \frac{\sin\left(\frac{t_j W_j \sqrt{pn}}{2}\right)}{\sin\left(\frac{t_j}{2}\right)}.$$

If $\frac{5}{W_j\sqrt{pn}} \leq |t_j| \leq \pi$ for some j , this quantity is at most $4/5$, so

$$|\mathbb{E}[e^{it \cdot S'}]| \leq (1 - C_3/5)^m = O(n^{-\omega(1)}).$$

If $n^{0.01/d}/\sqrt{n^2 p} < t_j < \frac{5}{W_j\sqrt{pn}}$ for some j , then this quantity is at most

$$\frac{1}{W_j\sqrt{pn}} \frac{\frac{t_j W_j \sqrt{pn}}{2} \left(1 - 0.01 t_j^2 W_j^2 p n\right)}{t_j/2 + O(t_j^3)} \leq 1 - n^{0.01/d}/n,$$

so

$$|\mathbb{E}[e^{it \cdot S'}]| \leq (1 - n^{0.01/d}/n)^m = O(n^{-\omega(1)}).$$

Thus, we have

$$\mathbb{P}[S' = 0] = \Omega((n^2 p)^{-d/2}) + \frac{1}{(2\pi)^d} \int_{\substack{t \in [-\pi, \pi]^d \\ \|t\|_\infty > n^{0.01/d}/\sqrt{n^2 p}}} dt \mathbb{E}[e^{it \cdot S'}] = \Omega((n^2 p)^{-d/2}),$$

as desired. $\square$

Theorem A.4. Fix a real parameter $\theta \in (1/2, 1)$, a positive integer $d \geq 1$, an integer matrix $M \in \mathbb{Z}^{r \times d}$ with pairwise orthogonal rows, an r -dimensional vector inequality relation $\geq' \in \{\geq, >\}^r$, a monomial $\rho \in \mathbb{R}[x_1, \dots, x_d]$, and a real constant $T > 1$. For all sufficiently large $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^{-\theta})$, the following is true.

Select $\tilde{\eta}, \eta \in ((T^{-1}n, Tn) \cap \mathbb{Z})^d$ such that $M\tilde{\eta} = 0$ and $\|\eta - \tilde{\eta}\|_\infty < T\frac{n}{\sqrt{pn}}$, and $\tilde{q}, q \in (0, 1)^d$ such that $\|\tilde{q} - p\|, \|q - p\| < T\frac{p}{\sqrt{pn}}$ and $\|\beta\|_\infty < T\frac{1}{\sqrt{pn} \log^2 n}$ where

$$\beta := \log \left(\frac{\frac{q}{1-q}(\eta - p\tilde{\eta})}{\frac{\tilde{q}}{1-\tilde{q}}(\tilde{\eta} - p\tilde{\eta})} \right).$$

Then,

$$\begin{aligned} \mathbb{E}[\rho(A - p\tilde{\eta})\mathbf{1}_{MA \geq' 0}] &= \mathbb{E}[\rho(\tilde{A} - p\tilde{\eta})\mathbf{1}_{M\tilde{A} \geq' 0}] \\ &\quad + \sum_{t \in [d]} \beta_t \mathbb{E}[(\tilde{A}_t - p\tilde{\eta}_t)\rho(\tilde{A} - p\tilde{\eta})\mathbf{1}_{M\tilde{A} \geq' 0}] - \sum_{t \in [d]} \beta_t \mathbb{E}[\tilde{A}_t - p\tilde{\eta}_t] \mathbb{E}[\rho(\tilde{A} - p\tilde{\eta})\mathbf{1}_{M\tilde{A} \geq' 0}] \\ &\quad + O(\max\{p, \|\beta\|_\infty^2 pn \log^2 n\} \sqrt{pn}^{\deg \rho} \log^{2+\deg \rho} n) \end{aligned}$$

where $A_t \sim \text{Bin}(\eta_t, q_t)$ and $\tilde{A}_t \sim \text{Bin}(\tilde{\eta}_t, \tilde{q}_t)$ are all sampled independently.

Proof. Note that by a standard Chernoff bound,

$$\mathbb{P}[|A_t - p\tilde{\eta}_t| > \sqrt{p\tilde{\eta}_t \log \tilde{\eta}_t} \text{ for some } t] = O(n^{-\omega(1)}),$$

so

$$\mathbb{E}[\rho(A - p\tilde{\eta})\mathbf{1}_{MA \geq' 0}] = \mathbb{E}[\rho(A - p\tilde{\eta})\mathbf{1}_{MA \geq' 0} \mathbf{1}_{|A_t - p\tilde{\eta}_t| \leq \sqrt{p\tilde{\eta}_t \log \tilde{\eta}_t} \forall t}].$$

Thus, by [Lemma A.1](#), we have

$$\mathbb{E}[\rho(A - p\tilde{\eta})\mathbf{1}_{MA \geq' 0}]$$

$$\begin{aligned}
& \sum_{\substack{x \in \Lambda \\ Mx \geq '0}} \rho(x) \prod_{t \in [d]} \left[\left(\frac{q_t(\eta_t - p\tilde{\eta}_t)}{(1 - q_t)p\tilde{\eta}_t} \right)^{x_t} \frac{(p\tilde{\eta}_t)!(p\tilde{\eta}_t)^{x_t}}{(p\tilde{\eta}_t + x_t)!} \right] \\
&= \frac{\sum_{\substack{x \in \Lambda \\ Mx \geq '0}} \prod_{t \in [d]} \left[\left(\frac{q_t(\eta_t - p\tilde{\eta}_t)}{(1 - q_t)p\tilde{\eta}_t} \right)^{x_t} \frac{(p\tilde{\eta}_t)!(p\tilde{\eta}_t)^{x_t}}{(p\tilde{\eta}_t + x_t)!} \right]}{\sum_{x \in \Lambda} \prod_{t \in [d]} \left[\left(\frac{q_t(\eta_t - p\tilde{\eta}_t)}{(1 - q_t)p\tilde{\eta}_t} \right)^{x_t} \frac{(p\tilde{\eta}_t)!(p\tilde{\eta}_t)^{x_t}}{(p\tilde{\eta}_t + x_t)!} \right]} + O(p\sqrt{pn}^{\deg \rho} \log^{2+\deg \rho} n) \\
&= \frac{\sum_{\substack{x \in \Lambda \\ Mx \geq '0}} \rho(x) \exp \left(\sum_{t \in [d]} \beta_t x_t \right) \prod_{t \in [d]} \left[\left(\frac{\tilde{q}_t(\tilde{\eta}_t - p\tilde{\eta}_t)}{(1 - \tilde{q}_t)p\tilde{\eta}_t} \right)^{x_t} \frac{(p\tilde{\eta}_t)!(p\tilde{\eta}_t)^{x_t}}{(p\tilde{\eta}_t + x_t)!} \right]}{\sum_{x \in \Lambda} \exp \left(\sum_{t \in [d]} \beta_t x_t \right) \prod_{t \in [d]} \left[\left(\frac{\tilde{q}_t(\tilde{\eta}_t - p\tilde{\eta}_t)}{(1 - \tilde{q}_t)p\tilde{\eta}_t} \right)^{x_t} \frac{(p\tilde{\eta}_t)!(p\tilde{\eta}_t)^{x_t}}{(p\tilde{\eta}_t + x_t)!} \right]} + O(p\sqrt{pn}^{\deg \rho} \log^{2+\deg \rho} n) \\
&= \frac{\sum_{\substack{x \in \Lambda \\ Mx \geq '0}} \rho(x) \left[1 + \sum_{t \in [d]} \beta_t x_t \right] \prod_{t \in [d]} \left[\left(\frac{\tilde{q}_t(\tilde{\eta}_t - p\tilde{\eta}_t)}{(1 - \tilde{q}_t)p\tilde{\eta}_t} \right)^{x_t} \frac{(p\tilde{\eta}_t)!(p\tilde{\eta}_t)^{x_t}}{(p\tilde{\eta}_t + x_t)!} \right]}{\sum_{x \in \Lambda} \left[1 + \sum_{t \in [d]} \beta_t x_t \right] \prod_{t \in [d]} \left[\left(\frac{\tilde{q}_t(\tilde{\eta}_t - p\tilde{\eta}_t)}{(1 - \tilde{q}_t)p\tilde{\eta}_t} \right)^{x_t} \frac{(p\tilde{\eta}_t)!(p\tilde{\eta}_t)^{x_t}}{(p\tilde{\eta}_t + x_t)!} \right]} + O(\max\{p, \|\beta\|_\infty^2 pn \log^2 n\} \sqrt{pn}^{\deg \rho} \log^{2+\deg \rho} n) \\
&= \frac{\mathbb{E} \left[\left(\rho(\tilde{A} - p\tilde{\eta}) + \sum_{t \in [d]} \beta_t (\tilde{A}_t - p\tilde{\eta}_t) \rho(\tilde{A} - p\tilde{\eta}) \right) \mathbf{1}_{M\tilde{A} \geq '0} \right]}{1 + \mathbb{E} \left[\sum_{t \in [d]} \beta_t \tilde{A}_t \right]} + O(\max\{p, \|\beta\|_\infty^2 pn \log^2 n\} \sqrt{pn}^{\deg \rho} \log^{2+\deg \rho} n) \\
&= \mathbb{E} \left[\rho(\tilde{A} - p\tilde{\eta}) \mathbf{1}_{M\tilde{A} \geq '0} \right] + \sum_{t \in [d]} \beta_t \mathbb{E} \left[(\tilde{A}_t - p\tilde{\eta}_t) \rho(\tilde{A} - p\tilde{\eta}) \mathbf{1}_{M\tilde{A} \geq '0} \right] - \sum_{t \in [d]} \beta_t \mathbb{E}[\tilde{A}_t - p\tilde{\eta}_t] \mathbb{E} \left[\rho(\tilde{A} - p\tilde{\eta}) \mathbf{1}_{M\tilde{A} \geq '0} \right] \\
&\quad + O(\max\{p, \|\beta\|_\infty^2 pn \log^2 n\} \sqrt{pn}^{\deg \rho} \log^{2+\deg \rho} n),
\end{aligned}$$

where $\Lambda := \prod_{t \in [d]} (\mathbb{Z} - p\tilde{\eta}_t) \cap [-\sqrt{p\tilde{\eta}_t} \log \tilde{\eta}_t, \sqrt{p\tilde{\eta}_t} \log \tilde{\eta}_t]$. This completes the proof. $\square$

APPENDIX B. IDEALIZED PROCESS

B.1. Specialized Inverse Function Theorems. We begin with a strong notion of bijectivity.

Definition B.1. Fix two convex sets $A, B \subset \mathbb{R}^d$. Call a bijection $f : A \rightarrow B$ *strongly bijective* if f and f^{-1} are smooth and for all compact $C \subset B$, we have

$$\inf_{\substack{y, z \in C \\ y \neq z}} \frac{|f^{-1}(y) - f^{-1}(z)|}{|y - z|} > 0$$

and

$$\sup_{\substack{y, z \in C \\ y \neq z}} \frac{|f^{-1}(y) - f^{-1}(z)|}{|y - z|} < \infty.$$

As an example, note that any linear map with nonzero determinant is a strong bijection.

Theorem B.2. Fix a positive definite matrix $\Sigma \in \mathbb{R}^{d \times d}$. Define

$$\begin{aligned}
f : \mathbb{R}^d &\rightarrow \mathbb{R}_{>0}^d \\
\mu &\mapsto \mathbb{E}_{X \sim \mathcal{N}(\mu, \Sigma)}[X | X > 0].
\end{aligned}$$

Then, f is a strong bijection.

Proof. We begin by transforming the problem to a standard Gaussian setting, at the cost of a more complex constraint condition. In particular, write $\Sigma = \Lambda\Lambda^\top$ for some invertible matrix $\Lambda \in \mathbb{R}^{d \times d}$. Note that $f(\mu) = \Lambda g(\Lambda^{-1}\mu)$ where

$$g: \mathbb{R}^d \rightarrow \Omega$$

$$\nu \mapsto \mathbb{E}_{X \sim \mathcal{N}(\nu, \text{id}_d)}[X | X \in \Omega]$$

where

$$\Omega := \{x \in \mathbb{R}^d : \Lambda x > 0\}.$$

It suffices to show that g is a strong bijection from $\mathbb{R}^d$ to Ω .

Note that

$$g(\nu) = \frac{\int_{x \in \Omega} dx e^{-|x-\nu|^2/2}}{\int_{x \in \Omega} dx e^{-|x-\nu|^2/2}},$$

which shows that $g \in C^\infty(\mathbb{R}^d)$.

We first show that g is a surjection, which is the most difficult part and which will form the bulk of the proof. Note that Ω is an infinite open polytope, and consider its boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$, where $\overline{\Omega}$ is the closure of Ω . Additionally, let $\omega_i = \Lambda^{-1}e_i$ for $1 \leq i \leq d$, where e_i are the elementary vectors. Thus, $\Omega = \text{span}_{\mathbb{R}_{>0}}\{\omega_1, \dots, \omega_d\}$ (i.e., the span where *all* coordinates must be strictly positive). For $1 \leq i \leq d$, let ω_i^* be the unit vector orthogonal to all ω_j for $j \neq i$ such that $\alpha_i := \omega_i^* \cdot \omega_i > 0$. Note that

$$\Omega = \bigcap_{j \in [d]} \{\nu \in \mathbb{R}^d : \nu \cdot \omega_j^* > 0 \text{ for all } j \in [d]\}.$$

We have the following estimate for g evaluated at points outside Ω .

Lemma B.3. *For all $\varepsilon > 0$, there exists $C_1(\varepsilon) > 0$ such that for all $\nu \in \mathbb{R}^d$ satisfying $d(\nu, \Omega) > C_1(\varepsilon)$, there exists $j \in [d]$ such that $\nu \cdot \omega_j^* < 0$ and $g(\nu) \cdot \omega_j^* < \varepsilon$.*

Proof. Find the unique $z \in \mathbb{R}^d$ such that $z \cdot \omega_j^* = 1$ for all $j \in [d]$. For any $\nu \in \mathbb{R}^d$, let

$$\Delta_\nu := -\min_{j \in [d]} \{\nu \cdot \omega_j^*\}.$$

Note that $\Delta_\nu > 0$ is equivalent to $\nu \notin \overline{\Omega}$. If $\Delta_\nu > 0$, then note that $(\nu + 2\Delta_\nu z) \cdot \omega_j^* > 0$ for all $j \in [d]$, so $\nu + 2\Delta_\nu z \in \Omega$, so $d(\nu, \Omega) \leq 2|z|\Delta_\nu$.

We have the following estimate for the distance of $g(\nu)$ to one of the bounding hyperplanes of Ω .

Claim. *There exists a constant $A > 0$ depending only on Ω such that for any $\nu \in \mathbb{R}^d$ with $\Delta_\nu > 0$, there exists an index $j \in [d]$ such that $\nu \cdot \omega_j^* = -\Delta_\nu$ and*

$$g(\nu) \cdot \omega_j^* \leq A\rho(\Delta_\nu)$$

where $\rho: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is defined as

$$\rho(t) := \frac{\int_0^\infty dh e^{-(h+t)^2/2} h}{\int_0^\infty dh e^{-(h+t)^2/2}}.$$

Proof. Consider an index $j \in [d]$ such that $\nu \cdot \omega_j^* = -\Delta_\nu$, which exists by definition. For every $h > 0$, let

$$V_h := \{y \in \mathbb{R}^d : y \cdot \omega_j^* = h\}.$$

Let $d_h x$ be the $(d-1)$ -dimensional Euclidean measure on V_h , and let ν_h be the orthogonal projection of ν onto V_h , so that $\nu_h = \nu_0 + h\omega_j^*$. We see that

$$g(\nu) \cdot \omega_j^* = \frac{\int_0^\infty dh e^{-(h+\Delta_\nu)^2/2} h \int_{x \in V_h \cap \Omega} d_h x e^{-|x-\nu_h|^2/2}}{\int_0^\infty dh e^{-(h+\Delta_\nu)^2/2} \int_{x \in V_h \cap \Omega} d_h x e^{-|x-\nu_h|^2/2}}.$$

The key observation is that $V_h \cap \Omega$ is a translated copy of $V_0 := \text{span}_{\mathbb{R}_{>0}}\{\omega_1, \dots, \omega_d\} \setminus \{\omega_j\}$. Quantitatively, since $\alpha_j = \omega_j^* \cdot \omega_j > 0$ we have

$$V_h \cap \Omega = (V_0 \cap \Omega) + h\alpha_j^{-1}\omega_j.$$

Thus, with the substitution $x = y + h\alpha_j^{-1}\omega_j$ we have

$$\sigma(h) := \int_{x \in V_h \cap \Omega} d_h x e^{-|x - \nu_h|^2/2} = \int_{y \in V_0 \cap \Omega} d_0 y e^{-|y - \nu_0 - h(\omega_j^* - \alpha_j^{-1}\omega_j)|^2/2},$$

where we remark that $\omega_j^* - \alpha_j^{-1}\omega_j \in V_0$.

Now, we have $\sigma(h) \leq (\sqrt{2\pi})^{d-1}$ for all h and

$$\sigma(h) \geq 1/B$$

for all $h \in [0, 1]$, where B is a positive absolute constant depending only on Ω . The former estimate derives from extending the Gaussian integral to infinity. The latter estimate is derived as follows. Given $h \in [0, 1]$, let $\tau = \nu_0 + h(\omega_j^* - \alpha_j^{-1}\omega_j) \in V_0$. We have $d(\tau, V_0 \cap \Omega) \leq C$, for appropriately chosen constant C . Then

$$\sigma(h) \geq \inf \left\{ \int_{y \in V_0 \cap \Omega} d_0 y e^{-|y - \tau|^2/2} : \tau \in V_0, d(\tau, V_0 \cap \Omega) \leq C \right\}.$$

This is the minimum of a positive continuous function (which only depends on Ω and the choice of one of d values for index j) over a compact region, which must be positive.

Now,

$$g(\nu) \cdot \omega_j^* = \frac{\int_0^\infty dh e^{-(h+\Delta_\nu)^2/2} h \sigma(h)}{\int_0^\infty dh e^{-(h+\Delta_\nu)^2/2} \sigma(h)} \leq \frac{(\sqrt{2\pi})^d \int_0^\infty dh e^{-(h+\Delta_\nu)^2/2} h}{\int_0^1 dh e^{-(h+\Delta_\nu)^2/2} / B} \leq A\rho(\Delta_\nu)$$

for appropriate constant A . The last inequality uses that

$$t \mapsto \frac{\int_0^1 dh e^{-(h+t)^2/2}}{\int_0^\infty dh e^{-(h+t)^2/2}}$$

is bounded from below by an absolute constant for $t \geq 0$. □

Thus, if $\Delta_\nu > 0$, choose the index j as above. Since ρ is a strictly decreasing function, we thus find that for all $\nu \notin \overline{\Omega}$ (i.e. $\nu \in \mathbb{R}^d$ with $\Delta_\nu > 0$),

$$g(\nu) \cdot \omega_j^* \leq A\rho\left(\frac{d(\nu, \Omega)}{2|z|}\right).$$

This completes the proof, as $\lim_{t \rightarrow \infty} \rho(t) = 0$. □

We have the following estimate for g evaluated at points close to Ω .

Lemma B.4. *For all $C > 0$, there exists $K_2(C)$ such that for all $\nu \in \mathbb{R}^d$ satisfying $d(\nu, \Omega) \leq C$, we have $|g(\nu) - \nu| < K_2(C)$.*

Proof. Let θ be the fractional dihedral angle spanned by Ω at the origin. Formally, define

$$\theta := \frac{\text{vol}_{d-1}(S^{d-1} \cap \Omega)}{\text{vol}_{d-1}(S^{d-1})}.$$

By definition, for any $z \in \overline{\Omega}$, we have $z + \Omega \subset \Omega$. For any $\nu \in \mathbb{R}^d$, letting γ be the closest point in $\overline{\Omega}$ to ν , the ball $B(\nu, 2d(\nu, \Omega))$ contains the ball $B(\gamma, d(\nu, \Omega))$. Thus

$$\text{vol}_d(\Omega \cap B(\nu, 2d(\nu, \Omega))) \geq \text{vol}_d((\gamma + \Omega) \cap B(\gamma, d(\nu, \Omega))) = \frac{\theta}{2^d} \text{vol}_d(B(0, 2d(\nu, \Omega))).$$

Thus, for all $\nu \in \mathbb{R}^d$ with $d(\nu, \Omega) \leq C$ we must have

$$\int_{x \in \Omega} dx e^{-|x-\nu|^2/2} \geq e^{-2d(\nu, \Omega)^2} \cdot \frac{\theta}{8} \text{vol}_d(B(0, d(\nu, \Omega))) \geq A e^{-2C^2} C^d$$

for some constant A depending only on d, Ω (and in particular independent of ν, C).

Thus, for any $r > 0$, we have

$$|g(\nu) - \nu| \leq \frac{\int_{x \in \Omega} dx e^{-|x-\nu|^2/2} |x - \nu|}{\int_{x \in \Omega} dx e^{-|x-\nu|^2/2}} \leq r + A^{-1} e^{2C^2} C^{-d} \int_{\substack{y \in \mathbb{R}^d \\ |y| > r}} dy e^{-|y|^2/2} |y|,$$

where we have used the substitution $y = x - \nu$. Note that the function $\sigma: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ given by

$$\sigma(r) = \int_{\substack{|y| > r \\ y \in \mathbb{R}^d}} |y| e^{-|y|^2/2} dy$$

tends to 0 as $r \rightarrow \infty$. Select $R(C) > 0$ such that $\sigma(R) < \frac{C^d A}{e^{2C^2}}$, so that taking $r = R(C)$ we have

$$|g(\nu) - \nu| < R(C) + 1.$$

Letting $K_2(C) = R(C) + 1$ completes the proof. $\square$

The surjection will now follow from the continuity of g combined with the estimates [Lemmas B.3](#) and [B.4](#). Assume for the sake of contradiction that there is some $y \in \Omega$ such that $y \notin g(\mathbb{R}^d)$.

Define

$$Q := \partial \left(\Lambda^{-1}[-1, 1]^d \right),$$

which is the surface of some parallelepiped that has faces parallel to those of Ω . The structure of the proof is to construct a homotopy between the constant map $Q \rightarrow \text{point}$ and the scaling map $Q \rightarrow \alpha Q$ for some large α that never crosses over y , which is a contradiction, since for large enough α , the second map encloses y , whereas the first map doesn't (formally we can look at the fundamental group of space punctured by y).

The main technical estimate we require is the following.

Lemma B.5. *For sufficiently large $\alpha > 0$ (in terms of y and Ω), the following is true. For all $x \in \alpha Q$, the line segment joining $g(x)$ and x does not contain y .*

Proof. Take $C = C_1(d(y, \partial\Omega)/2)$.

If $d(x, \Omega) > C$, then by [Lemma B.3](#), there exists $j \in [d]$ such that $x \cdot \omega_j^* < 0$ and $g(x) \cdot \omega_j^* < d(y, \partial\Omega)/2$. Therefore, for all $t \in [0, 1]$, we have

$$((1-t)x + tg(x)) \cdot \omega_j^* < d(y, \partial\Omega)/2.$$

However, note that $y \cdot \omega_j^* \geq d(y, \partial\Omega)$, since otherwise the intersection of the line segment joining y and the projection of y onto the hyperplane with normal vector ω_j^* (which is outside Ω) intersects $\partial\Omega$ at a point which is closer than $d(y, \partial\Omega)$ away from y , contradiction. Therefore, y is not on the line segment joining $g(x)$ and x .

Now suppose that $d(x, \Omega) \leq C$. Select α large enough such that $d(y, \alpha Q) > 2K_2(C)$ (which is possible since $d(y, \alpha Q) \rightarrow \infty$ as $\alpha \rightarrow \infty$). By [Lemma B.4](#), we have

$$|g(x) - x| < K_2(C),$$

and by construction we have $|x - y| > 2K_2(C)$. Thus, for all $t \in [0, 1]$, we have

$$|(1-t)x + tg(x) - y| \geq |x - y| - t|x - g(x)| > K_2(C),$$

so the line segment joining $g(x)$ and x does not contain y . $\square$

We now construct the problematic homotopy through a sequence of intermediate homotopies. Select α large according to [Lemma B.5](#), and sufficiently large so that y is in the interior of αQ . First, note that the maps $Q \rightarrow \text{point}$ and $Q \rightarrow g(\alpha Q)$ are homotopic through the homotopy

$$H: [0, 1] \times Q \rightarrow \mathbb{R}^d$$

$$(t, q) \mapsto g(tq),$$

which doesn't cross y since y was assumed to not be in the image of g . Second, the maps $Q \rightarrow g(\alpha Q)$ and $Q \rightarrow \alpha Q$ are homotopic through the homotopy

$$H: [0, 1] \times Q \rightarrow \mathbb{R}^d$$

$$(t, q) \mapsto (1 - t)g(\alpha q) + t\alpha q,$$

which doesn't cross y due to [Lemma B.5](#).

Putting these two homotopies together, we have a homotopy from the constant map $Q \rightarrow \text{point}$ to the map $Q \rightarrow \alpha Q$ that never crosses y . However, it is standard that the latter is nontrivial in the homotopy group of $\mathbb{R}^d \setminus y$. This is the desired contradiction, so y must have been in the image of g , as desired.

We now show that g is an injection, and that it has an invertible Jacobian everywhere. Direct computation on the integral formula for g tells us that the Jacobian is given by

$$\mathcal{J}_g(\nu)_{ab} = \frac{\partial g(\nu)_a}{\partial \nu_b} = \mathbb{E}_{X \sim \mathcal{N}(\nu, \text{id}_d)}[X_a X_b | X \in \Omega] - \mathbb{E}_{X \sim \mathcal{N}(\nu, \text{id}_d)}[X_a | X \in \Omega] \mathbb{E}_{X \sim \mathcal{N}(\nu, \text{id}_d)}[X_b | X \in \Omega].$$

For any direction $\eta \in \mathbb{R}^d$, we have

$$\eta^\top \mathcal{J}_g(\nu) \eta = \text{Var}_{X \sim \mathcal{N}(\nu, \text{id}_d)}[\eta \cdot X | X \in \Omega] > 0,$$

where the variance is positive since $\eta \cdot X$ is not constant almost everywhere (since Ω contains a cube where the pdf of X is bounded below by a positive constant everywhere). This shows that $\mathcal{J}_g(\nu)$ is invertible, and furthermore, that

$$t \mapsto \eta \cdot g(\nu + t\eta)$$

is a strictly increasing function, which directly implies that g is injective.

We now show that g^{-1} is smooth. Fix $x \in \mathbb{R}^d$. Since $\mathcal{J}_g(x)$ is invertible, by the inverse function theorem there exists an open neighborhood U_x of x such that $g^{-1}: g(U_x) \rightarrow U_x$ is smooth. This shows that g^{-1} is smooth at x , as desired.

Finally, the last remaining condition of a strong bijection is equivalent to showing that for every $N > 0$, there exists $C(N) > 0$ such that for all $y, z \in [-N, N]^d$, we have

$$\frac{1}{C(N)}|y - z| < |g(y) - g(z)| < C(N)|y - z|.$$

Consider the map

$$j: S^{d-1} \times [-N, N]^d \rightarrow \mathbb{R},$$

$$(u, x) \mapsto |\mathcal{J}_g(x)u|.$$

This is a continuous map on a compact set, so there exists $J(N) > 0$ such that $j(u, x) < J(N)$ for all $u \in S^{d-1}$ and $x \in [-N, N]^d$. Therefore,

$$|g(y) - g(z)| = \left| \int_0^1 dt \mathcal{J}_g(z + t(y - z))(y - z) \right| \leq J(N)|y - z|$$

by the fundamental theorem of calculus and the triangle inequality. For the lower bound, we can perform an analogous argument on g^{-1} . This completes the proof of the theorem. $\square$

Theorem B.6. Fix a positive integer $d \geq 1$ and an integer matrix $M \in \mathbb{Z}^{r \times d}$ with pairwise orthogonal rows. Let $\mathcal{D} = \{x \in \mathbb{R}^d : Mx > 0\}$ and fix a strong bijection $f : \mathbb{R}^d \rightarrow \mathcal{D}$.

Consider a family of smooth maps $f_\varepsilon : \mathbb{R}^d \rightarrow \mathcal{D}$ parameterized by $\varepsilon \in (0, \varepsilon_{\max})$ for some fixed $\varepsilon_{\max} > 0$ such that the following is true. For all $T > 0$, there exists $C(T) > 0$ such that for all $\varepsilon \in (0, \varepsilon_{\max})$ and $|x| < T$ we have $|f_\varepsilon(x) - f(x)| \leq C(T)\varepsilon$.

Fix a constant $T > 1$. There exists $\varepsilon_0, C > 0$ such that for all $\varepsilon < \varepsilon_0$, the following is true. For all $y \in \mathcal{D}$ with $\|My\|_\infty \geq T^{-1}$ and $\|y\|_\infty \leq T$, there exists $x \in \mathbb{R}^d$ such that $f_\varepsilon(x) = y$ and $|x - f^{-1}(y)| \leq C\varepsilon$.

Proof. For any constant $T' > 1$, define

$$\mathcal{D}_{T'} := \{y \in \mathcal{D} : \|y\|_\infty \leq T, \|My\|_\infty \geq T^{-1}\}.$$

Define

$$r := \frac{1}{2} \inf_{y \in \mathcal{D}_T} d(y, \partial \mathcal{D}),$$

which is some positive constant by compactness. For any $y \in \mathcal{D}$, let

$$\gamma_y : S^{d-1} \rightarrow \mathbb{R}^d$$

be a sphere centered at y with radius r . There exists a constant T' such that for all $y \in \mathcal{D}_T$, γ_y lies fully inside $\mathcal{D}_{T'}$.

Define

$$\alpha := \inf_{\substack{y, z \in \mathcal{D}_{T'} \\ y \neq z}} \frac{|f^{-1}(y) - f^{-1}(z)|}{|y - z|},$$

which by strong bijectivity is some positive constant. Note that all the points in the image of $f^{-1} \circ \gamma_y$ are at least αr away from $f^{-1}(y)$, for all $y \in \mathcal{D}_T$.

For all $\varepsilon \in (0, \varepsilon_{\max})$ and $y \in \mathcal{D}_T$, consider the homotopy

$$\begin{aligned} F_{\varepsilon, y} : [0, 1] \times S^{d-1} &\rightarrow \mathbb{R}^d \\ (t, \omega) &\mapsto ((1-t)f + tf_\varepsilon) \circ f^{-1} \circ \gamma_y(\omega) \end{aligned}$$

between γ_y and $f_\varepsilon \circ f^{-1} \circ \gamma_y$.

Note that

$$F_{\varepsilon, y}(t, \omega) = (1-t)\gamma_y(\omega) + tf_\varepsilon \circ f^{-1} \circ \gamma_y(\omega).$$

We see that $|f_\varepsilon \circ f^{-1} \circ \gamma_y(\omega) - \gamma_y(\omega)| \leq C(\alpha r)\varepsilon$, so in fact

$$|F_{\varepsilon, y}(t, \omega) - \gamma_y(\omega)| \leq C(\alpha r)\varepsilon.$$

Therefore, for $\varepsilon < \frac{r}{C(\alpha r)}$, we have that $F_{\varepsilon, y}(t, \omega) \neq y$, since $|\gamma_y(\omega) - y| = r$.

Now, consider a homotopy

$$\begin{aligned} G_{\varepsilon, y} : [0, 1] \times S^{d-1} &\rightarrow \mathbb{R}^d \\ (t, \omega) &\rightarrow f_\varepsilon((1-t)f^{-1} \circ \gamma_y(\omega)) \end{aligned}$$

which contracts $f_\varepsilon \circ f^{-1} \circ \gamma_y$ to the point $f(0)$.

Thus, the composition of the homotopies $F_{\varepsilon, y}$ and $G_{\varepsilon, y}$ produce a homotopy between γ_y and a point. Since the homotopy $F_{\varepsilon, y}$ never crosses y , we must have that the second homotopy crosses y . Thus, there is some point $x_\varepsilon(y) = (1-t)f^{-1} \circ \gamma_y(\omega)$ such that $f_\varepsilon(x_\varepsilon(y)) = y$. Note that by compactness, there is some constant T_1 such that $|x_\varepsilon(y)| < T_1$ for all $y \in \mathcal{D}_T$ and $\varepsilon < \frac{r}{C(\alpha r)}$. Thus, there is some constant T_2 such that $f(x_\varepsilon(y)) \in \mathcal{D}_{T_2}$ for all $y \in \mathcal{D}_T$ and $\varepsilon < \frac{r}{C(\alpha r)}$.

Define

$$\beta := \sup_{\substack{y, z \in \mathcal{D}_{T_2} \\ y \neq z}} \frac{|f^{-1}(y) - f^{-1}(z)|}{|y - z|},$$

which is finite by strong bijectivity. We now find that

$$|f^{-1}(y) - x| \leq \beta |f_\varepsilon(x_\varepsilon(y)) - f(x_\varepsilon(y))| \leq \beta C(T_2)\varepsilon.$$

Thus, selecting $\varepsilon_0 = \frac{r}{C(\alpha r)}$ and $C = C(T_2)$ completes the proof. $\square$

B.2. Universal Real Processes. We will define a sequence of real numbers parameterized by binary strings and pairs of binary strings. These sequences give universal estimates for the part sizes at various orders of approximation.

Definition B.7. Define a pair of coupled real-number sequences $(\nu_s)_{\cup_{k \geq 1} \{0,1\}^k}, (\mu_{s,t})_{s,t \in \cup_{k \geq 1} \{0,1\}^k}$. The definition is inductive in k with initial conditions are $\nu_0 = \nu_1 = 1/2$, and $\mu_{0,0} = \mu_{0,1} = \mu_{1,0} = \mu_{1,1} = 0$. The evolution rule $k \rightarrow k+1$ is as follows.

Pick the unique $\gamma \in \mathbb{R}^{\{0,1\}^k \times \{0,1\}^k}$ such that

$$\mathbb{E}[W_{s,t} | \mathcal{I}_s(W_{s,\bullet})] = \nu_t \mu_{s,t}$$

where $W_{s,t} \sim \mathcal{N}(\gamma_{s,t}, \nu_t)$ are all sampled independently. Then, for all $s, t \in \{0,1\}^k$ and $b, c \in \{0,1\}$, let

$$\nu_{sb} = \nu_s \mathbb{P}[\mathcal{J}_{sb}(W_{s,\bullet}) | \mathcal{I}_s(W_{s,\bullet})]$$

and

$$\mu_{sb,tc} = \frac{1}{\nu_t} \mathbb{E}[W_{s,t} | \mathcal{J}_{sb}(W_{s,\bullet})] + \frac{1}{\nu_s} \mathbb{E}[W_{t,s} | \mathcal{J}_{tc}(W_{t,\bullet})] - \mu_{s,t}.$$

Proof that Definition B.7 makes sense. Assuming the sequences exist till day k , it suffices to show that existence of a unique $\gamma \in \mathbb{R}^{\{0,1\}^k \times \{0,1\}^k}$ such that

$$\mathbb{E}[W_{s,t} | \mathcal{I}_s(W_{s,\bullet})] = \nu_t \mu_{s,t}$$

where $W_{s,t} \sim \mathcal{N}(\gamma_{s,t}, \nu_t)$ are all sampled independently. By Theorem B.2, it suffices to show that

$$\sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} \nu_t \mu_{s,t} \geq_{\bar{s}_r s_r} 0$$

for all $s, t \in \{0,1\}^k$ and $1 \leq r \leq k-1$.

We show this inductively in k , so it suffices to show

$$\sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} (\nu_{t0} \mu_{sb,t0} + \nu_{t1} \mu_{sb,t1}) \geq_{\bar{s}_r s_r} 0 \quad \text{for all } 1 \leq r \leq k-1$$

$$\sum_{t \in \{0,1\}^k} (-1)^{t_{k-1}} (\nu_{t0} \mu_{sb,t0} + \nu_{t1} \mu_{sb,t1}) \geq_{\bar{b}b} 0$$

for all $s, t \in \{0,1\}^k$ and $b \in \{0,1\}$. Indeed, we compute

$$\begin{aligned} \nu_{t0} \mu_{sb,t0} + \nu_{t1} \mu_{sb,t1} &= \mathbb{E}[W_{s,t} | \mathcal{J}_{sb}(W_{s,\bullet})] + \frac{\nu_t}{\nu_s} \left[\sum_{c \in \{0,1\}} \mathbb{P}[\mathcal{J}_{tc}(W_{t,s}) | \mathcal{I}_t(W_{t,\bullet})] \mathbb{E}[W_{t,s} | \mathcal{J}_{tc}(W_{t,\bullet})] \right] - \nu_t \mu_{s,t} \\ &= \mathbb{E}[W_{s,t} | \mathcal{J}_{sb}(W_{s,\bullet})] + \frac{\nu_t}{\nu_s} \mathbb{E}[W_{t,s} | \mathcal{I}_t(W_{t,\bullet})] - \nu_t \mu_{s,t} \\ &= \mathbb{E}[W_{s,t} | \mathcal{J}_{sb}(W_{s,\bullet})], \end{aligned}$$

where we have used $\mu_{s,t} = \mu_{t,s}$ in the last step (this result follows trivially by induction on k). This directly implies the result with the strict inequality replaced with a weak inequality. However, the

inequality is strict since $W_{s,\bullet}$ conditioned on $\mathcal{J}_{sb}(W_{s,\bullet})$ has support of full dimension, so the mean is strictly inside the region bounded by the $\mathcal{J}_{sb}$ inequalities. This completes the proof. $\square$

Definition B.8. Use the sequences ν, μ from [Definition B.7](#) as well as the auxiliary sequences γ, W . Define a real number sequence $(\varepsilon_s)_{s \in \{0,1\}^k}$. The definition is inductive in k with initial conditions $\varepsilon_0 = 1$ and $\varepsilon_1 = -1$. The evolution rule $k \rightarrow k+1$ is as follows.

Pick the unique $\beta \in \mathbb{R}^{\{0,1\}^k \times \{0,1\}^k}$ such that

$$\mathbb{E}[B_s W_{s,t} | \mathcal{I}_s(W_{s,\bullet})] - \mathbb{E}[B_s | \mathcal{I}_s(W_{s,\bullet})] \mathbb{E}[W_{s,t} | \mathcal{I}_s(W_{s,\bullet})] = \varepsilon_t$$

for all $s, t \in \{0,1\}^k$, where $B_s := \sum_{t' \in \{0,1\}^k} \beta_{s,t'} W_{s,t'}$. Then, for $s \in \{0,1\}^k$, $b \in \{0,1\}$, let

$$\varepsilon_{sb} = \nu_{sb} (\mathbb{E}[B_s | \mathcal{J}_{sb}(W_{s,\bullet})] - \mathbb{E}[B_s | \mathcal{I}_s(W_{s,\bullet})]).$$

Proof that [Definition B.8](#) makes sense. The only nontrivial part is existence and uniqueness of β . To show this, it suffices to show that for fixed $s \in \{0,1\}^k$, the matrix

$$(\mathbb{E}[W_{s,t} W_{s,t'} | \mathcal{I}_s(W_{s,\bullet})] - \mathbb{E}[W_{s,t} | \mathcal{I}_s(W_{s,\bullet})] \mathbb{E}[W_{s,t'} | \mathcal{I}_s(W_{s,\bullet})])_{t,t' \in \{0,1\}^k}$$

is invertible. Indeed, if it wasn't, then some nontrivial linear combination of $W_{s,\bullet}$ would be identically 0 when conditioned on $\mathcal{I}_s(W_{s,\bullet})$. This however is impossible since $W_{s,\bullet}$ conditioned on $\mathcal{I}_s(W_{s,\bullet})$ has support of full dimension, which completes the proof. $\square$

We have the following series of lemmas about the sequence $(\varepsilon_s)_{s \in \{0,1\}^k}$.

Lemma B.9. *For all positive integers k and $s \in \{0,1\}^k$, we have $\varepsilon_{s0} + \varepsilon_{s1} = 0$.*

Proof. Note that

$$\begin{aligned} \varepsilon_{s0} + \varepsilon_{s1} &= \nu_{s0} (\mathbb{E}[B_s | \mathcal{J}_{s0}(W_{s,\bullet})] - \mathbb{E}[B_s | \mathcal{I}_s(W_{s,\bullet})]) + \nu_{s1} (\mathbb{E}[B_s | \mathcal{J}_{s1}(W_{s,\bullet})] - \mathbb{E}[B_s | \mathcal{I}_s(W_{s,\bullet})]) \\ &= \nu_{s0} \mathbb{E}[B_s | \mathcal{J}_{s0}(W_{s,\bullet})] + \nu_{s1} \mathbb{E}[B_s | \mathcal{J}_{s1}(W_{s,\bullet})] - \nu_s \mathbb{E}[B_s | \mathcal{I}_s(W_{s,\bullet})] \\ &= \nu_s \mathbb{P}[\mathcal{J}_{s0}(W_{s,\bullet}) | \mathcal{I}(W_{s,\bullet})] \mathbb{E}[B_s | \mathcal{J}_{s0}(W_{s,\bullet})] + \nu_s \mathbb{P}[\mathcal{J}_{s1}(W_{s,\bullet}) | \mathcal{I}(W_{s,\bullet})] \mathbb{E}[B_s | \mathcal{J}_{s1}(W_{s,\bullet})] \\ &\quad - \nu_s \mathbb{E}[B_s | \mathcal{I}_s(W_{s,\bullet})] \\ &= 0, \end{aligned}$$

as desired. $\square$

Lemma B.10. *For all positive integers k and $s \in \{0,1\}^k$, we have $\varepsilon_{s0} > 0$ and $\varepsilon_{s1} < 0$.*

Proof. We proceed by induction on k with the $k=1$ case given by definition. Suppose the result is true for k , we will show concretely given fixed $s \in \{0,1\}^k$ that $\varepsilon_{s0} > 0$, which finishes by [Lemma B.9](#).

Fix $s \in \{0,1\}^k$. Consider the finite dimensional mean-translated Gaussian vector space

$$\mathcal{G} := \text{span}\{W_{s,t}\}_{t \in \{0,1\}^k},$$

where we recall that $W_{s,t} \sim \mathcal{N}(\gamma_{s,t}, \nu_t)$ are all sampled independently. Equip this vector space with inner product given by covariance, making it a Hilbert space.

For $1 \leq r \leq k-1$, let

$$Z_{s,r} = (-1)^{s_r} \left(\sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} W_{s,t} \right)$$

and let

$$Z_{s,k} = \sum_{t \in \{0,1\}^k} (-1)^{t_{k-1}} W_{s,t}.$$

Note that $Z_{s,1}, \dots, Z_{s,k}$ form a linearly independent set (because they are orthogonal with respect to the skewed inner product where all $W_{s,\bullet}$ have variance 1).

Up to an event of measure 0, the conditions $\mathcal{I}_s(W_{s,\bullet})$ are equivalent to $Z_{s,r} > 0$ for all $1 \leq r \leq k-1$, and $\mathcal{J}_{s0}(W_{s,\bullet})$ is equivalent to $Z_{s,r} > 0$ for all $1 \leq r \leq k$.

The definition of B_s found in [Definition B.8](#) combined with [Lemma B.9](#) implies that

$$\text{Cov}(B_s, Z_{s,r} | Z_{s,1}, \dots, Z_{s,k-1} > 0) = 0$$

for all $1 \leq r \leq k-1$. Decompose B_s by its orthogonal projection onto $\text{span}\{Z_{s,1}, \dots, Z_{s,k-1}\}$ (with respect to $\mathcal{G}$). Concretely, decompose

$$B_s = \sum_{r=1}^{k-1} \alpha_r Z_{s,r} + X$$

for some $\alpha_1, \dots, \alpha_{k-1} \in \mathbb{R}$ and $X \in \mathcal{G}$ where $\text{Cov}(X, Z_{s,r}) = 0$ for all $1 \leq r \leq k-1$. We see that X as a random variable is independent from the random variables $Z_{s,1}, \dots, Z_{s,k-1}$, so

$$\text{Cov}(X, Z_{s,r} | Z_{s,1}, \dots, Z_{s,k-1} > 0) = 0$$

for all $1 \leq r \leq k-1$. Therefore,

$$\sum_{r'=1}^{k-1} \alpha_{r'} \text{Cov}(Z_{s,r'}, Z_{s,r} | Z_{s,1}, \dots, Z_{s,k-1} > 0) = 0$$

for all $1 \leq r \leq k$. The covariance matrix

$$(\text{Cov}(Z_{s,r'}, Z_{s,r} | Z_{s,1}, \dots, Z_{s,k-1} > 0))_{r',r \in [k-1]}$$

is invertible since $Z_{s,1}, \dots, Z_{s,k-1}$ are linearly independent, so this implies that $\alpha_r = 0$ for all $1 \leq r \leq k-1$, so $B_s = X$.

The definition of B_s found in [Definition B.8](#) combined with the inductive hypothesis implies that

$$\text{Cov}(X, Z_{s,r} | Z_{s,1}, \dots, Z_{s,k-1} > 0) > 0.$$

Decompose $Z_{s,k}$ by its orthogonal projection onto $\text{span}\{Z_{s,1}, \dots, Z_{s,k-1}\}$. Concretely, decompose

$$Z_{s,k} = \sum_{r=1}^{k-1} \delta_r Z_{s,r} + Y$$

where $\text{Cov}(Y, Z_{s,r}) = 0$ for all $1 \leq r \leq k-1$. Plugging this back in, we find $\text{Cov}(X, Y | Z_{s,1}, \dots, Z_{s,k-1} > 0) = 0$, but since both X and Y are independent from $Z_{s,1}, \dots, Z_{s,k-1}$ as random variables, we have $\text{Cov}(X, Y) > 0$. Decompose $X = \alpha Y + X'$ where $\alpha > 0$ and $\text{Cov}(X', Y) = 0$.

Given the definition of ε_{s0} in [Definition B.8](#), it suffices to show that

$$\mathbb{E}[X | Z_{s,1}, \dots, Z_{s,k} > 0] > \mathbb{E}[X | Z_{s,1}, \dots, Z_{s,k-1} > 0].$$

We see that

$$\begin{aligned} & \mathbb{E}[X | Z_{s,1}, \dots, Z_{s,k} > 0] - \mathbb{E}[X | Z_{s,1}, \dots, Z_{s,k-1} > 0] \\ &= \alpha (\mathbb{E}[Y | Z_{s,1}, \dots, Z_{s,k} > 0] - \mathbb{E}[Y | Z_{s,1}, \dots, Z_{s,k-1} > 0]) \\ &= \alpha (\mathbb{E}[Y | Z_{s,1}, \dots, Z_{s,k-1} > 0, Y > -(\delta_1 Z_{s,1} + \dots + \delta_{k-1} Z_{s,k-1})] - \mathbb{E}[Y]) \\ &= \alpha (\mathbb{E}[Y | Y > U] - \mathbb{E}[Y]) \end{aligned}$$

where U is the random variable $-(\delta_1 Z_{s,1} + \dots + \delta_{k-1} Z_{s,k-1})$ conditioned on $Z_{s,1}, \dots, Z_{s,k-1} > 0$. Note that Y and U are independent as random variables, so clearly $\mathbb{E}[Y | Y > U] > \mathbb{E}[Y]$, so

$$\mathbb{E}[X | Z_{s,1}, \dots, Z_{s,k} > 0] - \mathbb{E}[X | Z_{s,1}, \dots, Z_{s,k-1} > 0] > 0,$$

which completes the proof. □

B.3. Idealized Process. We now describe an idealized part size evolution that gives a really precise estimate of the true part size evolution.

Theorem B.11. *Fix a real parameter $\theta \in (1/2, 1)$, a positive integer $D \geq 1$, and a constant $T > 1$. There exists a constant $\ell \geq 1$ such that for all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^\theta)$, the following sequence of numbers $(\tilde{n}_s(N) \in \mathbb{Z})_{s \in \cup_{1 \leq k \leq D} \{0,1\}^k}$, $(\tilde{m}_{s,t}(N) \in \mathbb{R})_{s,t \in \cup_{1 \leq k \leq D} \{0,1\}^k}$, and $(\tilde{q}_{s,t}(N) \in (0,1))_{s,t \in \cup_{1 \leq k \leq D} \{0,1\}^k}$ all exist satisfying the following properties (suppressing the dependence on N).*

- We have base case conditions $\tilde{n}_0 = \tilde{n}_1 = \lfloor N/2 \rfloor$ and $\tilde{m}_{s,t} = p \lfloor N/2 \rfloor^2$ for all $s, t \in \{0,1\}$.
- For all $1 \leq k \leq D-1$, we have

$$\tilde{n}_s \mathbb{E}[\tilde{A}_{s,t} | \mathcal{I}_s(\tilde{A}_{s,\bullet})] = \tilde{m}_{s,t}$$

where $\tilde{A}_{s,t} \sim \text{Bin}(\tilde{n}_t, \tilde{q}_{s,t})$ are all sampled independently and $\tilde{q}_{s,t} = \tilde{q}_{\bar{s},\bar{t}}$ and

$$\tilde{q}_{s,t} = \frac{\frac{p}{1-p} e^{\frac{\gamma_{s,t}}{\nu_t \sqrt{pN}}}}{1 + \frac{p}{1-p} e^{\frac{\gamma_{s,t}}{\nu_t \sqrt{pN}}}} + O(\log^\ell N/N).$$

- The evolution rule is as follows. For $1 \leq k \leq D-1$, and for all $s, t \in \{0,1\}^k$ and $b, c \in \{0,1\}$, we have

$$\tilde{n}_{sb} = \lfloor \tilde{n}_s \mathbb{P}[\mathcal{J}_{sb}(\tilde{A}_{s,\bullet}) | \mathcal{I}_s(\tilde{A}_{s,\bullet})] \rfloor$$

and

$$\tilde{m}_{sb,tc} = \frac{\tilde{n}_s \tilde{n}_t \mathbb{E} \left[\tilde{A}_{s,t} \mathbf{1}_{\mathcal{J}_{sb}(\tilde{A}_{s,\bullet})} | \mathcal{I}_s(\tilde{A}_{s,\bullet}) \right] \mathbb{E} \left[\tilde{A}_{t,s} \mathbf{1}_{\mathcal{J}_{tc}(\tilde{A}_{t,\bullet})} | \mathcal{I}_t(\tilde{A}_{t,\bullet}) \right]}{\tilde{m}_{s,t}}.$$

- For all $1 \leq k \leq D$ and $s, t \in \{0,1\}^k$, we have the symmetry relation $\tilde{n}_s = \tilde{n}_{\bar{s}}$ and $\tilde{m}_{s,t} = \tilde{m}_{\bar{s},\bar{t}}$ and estimates $\tilde{n}_s = N\nu_s + O(\frac{N}{\sqrt{pN}} \log^\ell N)$ and $\tilde{m}_{s,t} = p\tilde{n}_s\tilde{n}_t(1 + \mu_{s,t}/\sqrt{pN} + O(\log^\ell N/pN))$.

Proof. We proceed by induction on D , with the base case $D = 1$ being immediate. Suppose the result is true with D replaced by D' for all $1 \leq D' < D$, with log exponent parameter ℓ .

We begin by showing the second bullet point for $k = D-1$. From the inductive hypothesis, we know that $\tilde{n}_s = \tilde{n}_{\bar{s}}$ and $\tilde{m}_{s,t} = \tilde{m}_{\bar{s},\bar{t}}$ for all $s, t \in \{0,1\}^k$, and $\tilde{n}_s = N\nu_s + O(\frac{N}{\sqrt{pN}} \log^\ell N)$ and $\tilde{m}_{s,t} = p\tilde{n}_s\tilde{n}_t(1 + \mu_{s,t}/\sqrt{pN} + O(\log^\ell N/pN))$. For all $s \in \{0,1\}^k$, let

$$\mathcal{D}_s := \left\{ x \in \mathbb{R}^{\{0,1\}^k} : \sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} x_t \geq_{\bar{s}_r s_r} 0 \quad \forall 1 \leq r \leq k-1 \right\}$$

and define the functions

$$f_{s,N} : \mathbb{R}^{\{0,1\}^k} \rightarrow \mathcal{D}_s$$

$$\sigma \mapsto \left(\frac{\mathbb{E}[A_t | \mathcal{I}_s(A)] - p\tilde{n}_t}{\sqrt{pN}} \right)_{t \in \{0,1\}^k}$$

where we sample $A_t \sim \text{Bin} \left(\tilde{n}_t, \frac{\frac{p}{1-p} e^{\frac{\sigma_t}{\nu_t \sqrt{pN}}}}{1 + \frac{p}{1-p} e^{\frac{\sigma_t}{\nu_t \sqrt{pN}}}} \right)$ for all $t \in \{0,1\}^k$ all independently, and

$$f_s : \mathbb{R}^{\{0,1\}^k} \rightarrow \mathcal{D}_s$$

$$\sigma \mapsto \mathbb{E}[W | \mathcal{I}_s(W)]$$

where we sample $W_t \sim \mathcal{N}(\sigma_t, \nu_t)$ for all $t \in \{0,1\}^k$ all independently.

[Theorem A.2](#) implies that for all $T' > 0$, there exists $C(T') > 0$ independent of N such that

$$|f_{s,N}(\sigma) - f_s(\sigma)| \leq \frac{\log^{2+2^k} N}{\sqrt{pN}} C(T')$$

for all $|\sigma| < T'$ and all sufficiently large N . Furthermore, [Theorem B.2](#) implies that f_s is a strong bijection.

Let $y^{(s)} \in \mathbb{R}^{\{0,1\}^k}$ be defined by $y_t^{(s)} = \frac{\tilde{m}_{s,t} - p\tilde{n}_t}{\sqrt{pN}}$. We have $y_t^{(s)} = \nu_t \mu_{s,t} + O(\frac{\log^\ell N}{\sqrt{pN}})$, which by the proof of [Definition B.7](#) implies that there is a constant T' such that $\|y^{(s)}\|_\infty < T'$ and $y^{(s)} \in \mathcal{D}_s$ and $\left\| \sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} y_t^{(s)} \right\|_\infty \geq T'^{-1}$ for all $1 \leq r \leq k-1$. Thus, by [Theorem B.6](#), there is some constant C such that there exists $\sigma_s \in \mathbb{R}^{\{0,1\}^k}$ with $f_{s,N}(\sigma_s) = y^{(s)}$ and $\|\sigma_s - f_s^{-1}(y^{(s)})\|_\infty \leq C \frac{\log^{2+2^k} N}{\sqrt{pN}}$.

Note that due to the bit flip symmetry present in the inductive hypothesis, we can select σ_s such that $(\sigma_s)_t = (\sigma_{\bar{s}})_{\bar{t}}$ for all $s, t \in \{0,1\}^k$. Now, select

$$\tilde{q}_{s,t} = \frac{\frac{p}{1-p} e^{\frac{(\sigma_s)_t}{\nu_t \sqrt{pN}}}}{1 + \frac{p}{1-p} e^{\frac{(\sigma_s)_t}{\nu_t \sqrt{pN}}}}.$$

The second bullet point now follows since $\|\gamma_{s,\bullet} - f_s^{-1}(y^{(s)})\|_\infty = O\left(\frac{\log^\ell N}{\sqrt{pN}}\right)$ (as $\gamma_{s,\bullet} = f_s^{-1}(\nu_\bullet \mu_{s,\bullet})$, and f^{-1} has bounded gradient in a compact set), so $\|\sigma_s - \gamma_{s,\bullet}\|_\infty = O\left(\frac{\log^{\max\{\ell, 2+2^k\}} N}{\sqrt{pN}}\right)$, so

$$\tilde{q}_{s,t} = \frac{\frac{p}{1-p} e^{\frac{\gamma_{s,t}}{\nu_t \sqrt{pN}}}}{1 + \frac{p}{1-p} e^{\frac{\gamma_{s,t}}{\nu_t \sqrt{pN}}}} + O(\log^{\max\{\ell, 2+2^k\}} N/N).$$

This shows that we can select $\tilde{q}$ for $k = D-1$ in accordance to the conditions of the second bullet point.

We now show that the fourth bullet point holds for $k = D$, assuming the evolution rule given by the third bullet point, which we can use given that we've shown the second bullet point. The symmetry relations are obvious by the symmetry of the inductive hypothesis and the symmetries in $\tilde{q}$ from the second bullet point we just showed. It suffices to prove the first order estimates in terms of the universal real number sequences constructed in [Definition B.7](#).

By [Theorem A.2](#), we have that there is some constant ℓ' such that

$$\mathbb{P}[\mathcal{J}_{sb}(\tilde{A}_{s,\bullet}) | \mathcal{I}_s(\tilde{A}_{s,\bullet})] = \mathbb{P}[\mathcal{J}_{sb}(W_{s,\bullet}) | \mathcal{I}_s(W_{s,\bullet})] + O(\log^{\ell'} N / \sqrt{pN}) = \frac{\nu_{sb}}{\nu_s} + O(\log^{\ell'} N / \sqrt{pN}),$$

recalling that $W_{s,t} \sim \mathcal{N}(\gamma_{s,t}, \nu_t)$, which implies that $\tilde{n}_{sb} = N\nu_{sb} + O(\frac{N}{\sqrt{pN}} \log^{\max\{\ell, \ell'\}} N)$.

Furthermore, [Theorem A.2](#) also implies that for some potentially different constant ℓ' ,

$$\begin{aligned} & \mathbb{E} \left[\tilde{A}_{s,t} \mathbf{1}_{\mathcal{J}_{sb}(\tilde{A}_{s,\bullet})} | \mathcal{I}_s(\tilde{A}_{s,\bullet}) \right] \\ &= p\tilde{n}_t \mathbb{P}[\mathcal{J}_{sb}(\tilde{A}_{s,\bullet}) | \mathcal{I}_s(\tilde{A}_{s,\bullet})] + \sqrt{pN} \mathbb{E} [W_{s,t} \mathbf{1}_{\mathcal{J}_{sb}(W_{s,\bullet})} | \mathcal{I}_s(W_{s,\bullet})] + O(\log^{\ell'} N) \\ &= p\tilde{n}_t \frac{\tilde{n}_{sb}}{\tilde{n}_s} + \sqrt{pN} \frac{\tilde{n}_{sb}}{\tilde{n}_s} \mathbb{E}[W_{s,t} | \mathcal{J}_{sb}(W_{s,\bullet})] + O(\log^{\ell'} N). \end{aligned}$$

Thus, we find

$$\tilde{m}_{sb,tc} = \frac{\tilde{n}_s \tilde{n}_t}{\tilde{m}_{s,t}} \left[p^2 \tilde{n}_{sb} \tilde{n}_{tc} + p \frac{\tilde{n}_{sb} \tilde{n}_{tc}}{\tilde{n}_s} \sqrt{pN} \mathbb{E}[W_{t,s} | \mathcal{J}_{tc}(W_{t,\bullet})] + p \frac{\tilde{n}_{sb} \tilde{n}_{tc}}{\tilde{n}_t} \sqrt{pN} \mathbb{E}[W_{s,t} | \mathcal{J}_{sb}(W_{s,\bullet})] + O(Np \log^{\ell'} N) \right]$$

$$\begin{aligned}
&= \frac{p\tilde{n}_s\tilde{n}_t}{\tilde{m}_{s,t}} p\tilde{n}_{sb}\tilde{n}_{tc} \left[1 + \frac{1}{\sqrt{pN}} \left[\frac{N}{\tilde{n}_s} \mathbb{E}[W_{t,s}|\mathcal{I}_{tc}(W_{t,\bullet})] + \frac{N}{\tilde{n}_t} \mathbb{E}[W_{s,t}|\mathcal{I}_{sb}(W_{s,\bullet})] \right] + O(\log^{\ell'} N/pN) \right] \\
&= p\tilde{n}_{sb}\tilde{n}_{tc} \left[1 + \frac{1}{\sqrt{pN}} \left[-\mu_{s,t} + \frac{N}{\tilde{n}_s} \mathbb{E}[W_{t,s}|\mathcal{I}_{tc}(W_{t,\bullet})] + \frac{N}{\tilde{n}_t} \mathbb{E}[W_{s,t}|\mathcal{I}_{sb}(W_{s,\bullet})] \right] + O(\log^{\max\{\ell', \ell\}} N/pN) \right] \\
&= p\tilde{n}_{sb}\tilde{n}_{tc} \left[1 + \frac{1}{\sqrt{pN}} \mu_{sb,tc} + O(\log^{\max\{\ell', \ell\}} N/pN) \right],
\end{aligned}$$

as desired. This completes the proof. $\square$

B.4. Perturbed Part Size Evolution. We now describe estimates for the part size evolution with part sizes perturbed off of the idealized values.

Theorem B.12. *Fix a real parameter $\theta \in (1/2, 1)$, a positive integer $k \in [1, \frac{1}{1-\theta})$ and constants $T > 1, \delta > 0$ and $\mathcal{T} > 1$. There exists constants $T' > 1, \delta' > 0$ such that for all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$ and $\tau \in [\mathcal{T}^{-1}, \mathcal{T}]$, the following statement is true.*

For all $n \in \mathbb{Z}^{\{0,1\}^k}$ and $m \in \mathbb{R}^{\{0,1\}^k \times \{0,1\}^k}$ satisfying

$$|n_s - \tilde{n}_s(N) - \tau\sqrt{N}\sqrt{pN}^{k-1}\varepsilon_s| \leq T\sqrt{N}\sqrt{pN}^{k-1}N^{-\delta}$$

and

$$\left| m_{s,t} - \tilde{m}_{s,t}(N) \left[1 + \tau \frac{\sqrt{pN}^{k-1}}{\sqrt{N}} \left(\frac{\varepsilon_s}{\nu_s} + \frac{\varepsilon_t}{\nu_t} \right) \right] \right| \leq T \frac{\sqrt{pN}^{k-1}}{\sqrt{N}} N^{-\delta} N^2 p,$$

there exists $q \in (0, 1)^{\{0,1\}^k \times \{0,1\}^k}$ satisfying

$$\left| \log \left[\frac{\frac{q_{s,t}}{1-q_{s,t}}(n_t - \mathbf{1}_{s=t} - p\tilde{n}_t(N))}{\frac{\tilde{q}_{s,t}(N)}{1-\tilde{q}_{s,t}(N)}(\tilde{n}_t(N) - p\tilde{n}_t(N))} \right] - \tau \frac{\sqrt{pN}^{k-1}}{\sqrt{N}} \beta_{s,t} \right| \leq T' \frac{\sqrt{pN}^{k-1}}{\sqrt{N}} N^{-\delta'}$$

such that

$$n_s \mathbb{E}[A_{s,t}|\mathcal{I}_s(A_{s,\bullet})] = m_{s,t}$$

where $A_{s,t} \sim \text{Bin}(n_s - \mathbf{1}_{s=t}, q_{s,t})$ are all sampled independently, and for all $s, t \in \{0, 1\}^k$ and $b, c \in \{0, 1\}$, we have

$$|n_{sb} - \tilde{n}_{sb}(N) - \tau\sqrt{N}\sqrt{pN}^k\varepsilon_{sb}| \leq T'\sqrt{N}\sqrt{pN}^kN^{-\delta'}$$

and

$$\left| m_{sb,tc} - \tilde{m}_{sb,tc}(N) \left[1 + \tau \frac{\sqrt{pN}^k}{\sqrt{N}} \left(\frac{\varepsilon_{sb}}{\nu_{sb}} + \frac{\varepsilon_{tc}}{\nu_{tc}} \right) \right] \right| \leq T' \frac{\sqrt{pN}^k}{\sqrt{N}} N^{-\delta'} N^2 p,$$

where

$$n_{sb} = n_s \mathbb{P}[\mathcal{I}_{sb}(A_{s,\bullet})|\mathcal{I}_s(A_{s,\bullet})]$$

and

$$m_{sb,tc} = \frac{n_s n_t \mathbb{E}[A_{s,t} \mathbf{1}_{\mathcal{I}_{sb}(A_{s,\bullet})} | \mathcal{I}_s(A_{s,\bullet})] \mathbb{E}[A_{t,s} \mathbf{1}_{\mathcal{I}_{tc}(A_{t,\bullet})} | \mathcal{I}_t(A_{t,\bullet})]}{m_{s,t}}.$$

Proof. Define the function

$$\begin{aligned}
f_{s,N} : \mathbb{R}^{\{0,1\}^k} &\rightarrow \mathbb{R}^{\{0,1\}^k} \\
\sigma &\mapsto \left(\frac{\mathbb{E}[A_t | \mathcal{I}_s(A)] - \frac{\tilde{m}_{s,t}}{\tilde{n}_s}}{\tau \frac{\sqrt{pN}^{k+1}}{\sqrt{N}}} \right)_{t \in \{0,1\}^k}
\end{aligned}$$

where we sample $A_t \sim \text{Bin} \left(n_t - \mathbf{1}_{s=t}, \frac{\frac{\tilde{n}_t - p\tilde{n}_t}{n_t - \mathbf{1}_{s=t} - p\tilde{n}_t} \frac{\tilde{q}_{s,t}}{1 - \tilde{q}_{s,t}} e^{\tau \frac{\sigma_t \sqrt{pN}^{k-1}}{\sqrt{N}}}}{1 + \frac{\tilde{n}_t - p\tilde{n}_t}{n_t - \mathbf{1}_{s=t} - p\tilde{n}_t} \frac{\tilde{q}_{s,t}}{1 - \tilde{q}_{s,t}} e^{\tau \frac{\sigma_t \sqrt{pN}^{k-1}}{\sqrt{N}}}} \right)$ all independently, and define the function

$$f_s : \mathbb{R}^{\{0,1\}^k} \rightarrow \mathbb{R}^{\{0,1\}^k}$$

$$\sigma \mapsto \left(\sum_{t' \in \{0,1\}^k} \sigma_{t'} (\mathbb{E}[W_{s,t'} W_{s,t} | \mathcal{I}_s(W_{s,\bullet})] - \mathbb{E}[W_{s,t'} | \mathcal{I}_s(W_{s,\bullet})] \mathbb{E}[W_{s,t} | \mathcal{I}_s(W_{s,\bullet})]) \right)_{t \in \{0,1\}^k}$$

where $W_{s,t} \sim \mathcal{N}(\gamma_{s,t}, \nu_t)$ are all sampled independently. We have the following relation between $f_{s,N}$ and f_s .

Lemma B.13. *There exists $\kappa > 0$ such that for all $R > 0$, there exists $C(R) > 0$ independent of N such that*

$$|f_{s,N}(\sigma) - f_s(\sigma)| \leq C(R)N^{-\kappa}$$

for all $|\sigma| < R$ and all sufficiently large N .

Proof. Fix $R > 0$ and $|\sigma| < R$. By [Theorem A.4](#), we have

$$\begin{aligned} \mathbb{E}[A_t | \mathcal{I}_s(A)] &= \mathbb{E}[\tilde{A}_{s,t} | \mathcal{I}_s(\tilde{A}_{s,\bullet})] \\ &+ \sum_{t' \in \{0,1\}^k} \tau \frac{\sigma_{t'} \sqrt{pN}^{k-1}}{\sqrt{N}} \left[\mathbb{E} \left[(\tilde{A}_{s,t'} - p\tilde{n}_{t'}) (\tilde{A}_{s,t} - p\tilde{n}_t) | \mathcal{I}_s(A_{s,\bullet}) \right] \right. \\ &\quad \left. - \mathbb{E} \left[\tilde{A}_{s,t'} - p\tilde{n}_{t'} | \mathcal{I}_s(A_{s,\bullet}) \right] \mathbb{E} \left[\tilde{A}_{s,t} - p\tilde{n}_t | \mathcal{I}_s(A_{s,\bullet}) \right] \right] \\ &+ O \left(\frac{(pN)^{k+1/2} \log^5 N}{N} \right) \\ &= \frac{\tilde{m}_{s,t}}{\tilde{n}_s} + \tau \frac{\sqrt{pN}^{k+1}}{\sqrt{N}} f_s(\sigma)_t + O \left(\frac{\sqrt{pN}^{k+1}}{\sqrt{N}} N^{-\kappa} \right), \end{aligned}$$

for some constant $\kappa > 0$ independent of N , where the second equality follows from [Theorem A.2](#). This immediately implies the desired result. $\square$

The proof of [Definition B.8](#) implies that f_s is a bijective linear map, so in particular it is a strong bijection.

Let $y^{(s)} \in \mathbb{R}^{\{0,1\}^k}$ be defined by

$$y_t^{(s)} := \frac{\frac{m_{s,t}}{n_s} - \frac{\tilde{m}_{s,t}}{\tilde{n}_s}}{\tau \frac{\sqrt{pN}^{k+1}}{\sqrt{N}}} = \varepsilon_t + O(N^{-\kappa'})$$

for some constant $\kappa' > 0$ independent of N . By [Theorem B.6](#), there is some constant $C > 0$ such that there exists $\sigma_s \in \mathbb{R}^{\{0,1\}^k}$ with $f_{s,N}(\sigma_s) = y^{(s)}$ and $\|\sigma_s - f_s^{-1}(y^{(s)})\|_\infty \leq CN^{-\kappa}$. Note that $\|f_s^{-1}(y^{(s)}) - f_s^{-1}(\varepsilon)\|_\infty = O(N^{-\kappa'})$ since f^{-1} has bounded gradient in a compact set. Remembering that $f_s^{-1}(\varepsilon) = \beta_{s,\bullet}$, we find that there is some potentially different constant $C > 0$ such that

$$\|\sigma_s - \beta_{s,\bullet}\|_\infty \leq CN^{-\min\{\kappa, \kappa'\}}.$$

Now, let

$$q_{s,t} = \frac{\frac{\tilde{n}_t - p\tilde{n}_t}{n_t - \mathbf{1}_{s=t} - p\tilde{n}_t} \frac{\tilde{q}_{s,t}}{1 - \tilde{q}_{s,t}} e^{\tau \frac{(\sigma_s)_t \sqrt{pN}^{k-1}}{\sqrt{N}}}}{1 + \frac{\tilde{n}_t - p\tilde{n}_t}{n_t - \mathbf{1}_{s=t} - p\tilde{n}_t} \frac{\tilde{q}_{s,t}}{1 - \tilde{q}_{s,t}} e^{\tau \frac{(\sigma_s)_t \sqrt{pN}^{k-1}}{\sqrt{N}}}},$$

so the existence of q satisfying the desired inequality and expectation equality is established. Now, letting $A_{s,t} \sim \text{Bin}(n_t - \mathbf{1}_{s=t}, q_{s,t})$ all be sampled independently, our goal is to provide the right estimates for n_{sb} and $m_{sb,tc}$.

By [Theorem A.4](#), we have

$$\begin{aligned} \mathbb{P}[\mathcal{J}_{sb}(A_{s,\bullet}) | \mathcal{I}_s(A_{s,\bullet})] &= \mathbb{P}[\mathcal{J}_{sb}(\tilde{A}_{s,\bullet}) | \mathcal{I}_s(\tilde{A}_{s,\bullet})] \\ &+ \sum_{t \in \{0,1\}^k} \tau \frac{\beta_{s,t} \sqrt{pN}^{k-1}}{\sqrt{N}} \left[\mathbb{E} \left[(\tilde{A}_{s,t} - p\tilde{n}_t) \mathbf{1}_{\mathcal{J}_{sb}(\tilde{A}_{s,\bullet})} | \mathcal{I}_s(\tilde{A}_{s,\bullet}) \right] \right. \\ &\quad \left. - \mathbb{E} \left[(\tilde{A}_{s,t} - p\tilde{n}_t) | \mathcal{I}_s(\tilde{A}_{s,\bullet}) \right] \mathbb{P} \left[\mathcal{J}_{sb}(\tilde{A}_{s,\bullet}) | \mathcal{I}_s(\tilde{A}_{s,\bullet}) \right] \right] \\ &+ O \left(\frac{\sqrt{pN}^k}{\sqrt{N}} N^{-\kappa} \right) \\ &= \mathbb{P}[\mathcal{J}_{sb}(\tilde{A}_{s,\bullet}) | \mathcal{I}_s(\tilde{A}_{s,\bullet})] + \tau \frac{\sqrt{pN}^k}{\sqrt{N}} \frac{\nu_{sb}}{\nu_s} \sum_{t \in \{0,1\}^k} \beta_{s,t} (\mathbb{E}[W_{s,t} | \mathcal{J}_{sb}(W_{s,\bullet})] - \mathbb{E}[W_{s,t} | \mathcal{I}_s(W_{s,\bullet})]) \\ &+ O \left(\frac{\sqrt{pN}^k}{\sqrt{N}} N^{-\kappa'} \right) \\ &= \mathbb{P}[\mathcal{J}_{sb}(\tilde{A}_{s,\bullet}) | \mathcal{I}_s(\tilde{A}_{s,\bullet})] + \tau \frac{\sqrt{pN}^k}{\sqrt{N}} \frac{\varepsilon_{sb}}{\nu_s} + O \left(\frac{\sqrt{pN}^k}{\sqrt{N}} N^{-\kappa'} \right) \end{aligned}$$

for some potentially different constants κ, κ' . This establishes the desired estimate for n_{sb} .

We now prove the desired estimate for $m_{sb,tc}$. We can rewrite

$$m_{sb,tc} = \frac{n_{sb} n_{tc} \mathbb{E}[A_{s,t} | \mathcal{J}_{sb}(A_{s,\bullet})] \mathbb{E}[A_{t,s} | \mathcal{J}_{tc}(A_{t,\bullet})]}{m_{s,t}}.$$

By [Theorem A.4](#), we have

$$\mathbb{E}[A_{s,t} | \mathcal{J}_{sb}(A_{s,\bullet})] = \mathbb{E}[\tilde{A}_{s,t} | \mathcal{J}_{sb}(\tilde{A}_{s,\bullet})] + O \left(\frac{\sqrt{pN}^{k+1}}{\sqrt{N}} \right),$$

so in fact

$$m_{sb,tc} = \frac{n_{sb} n_{tc} \mathbb{E}[\tilde{A}_{s,t} | \mathcal{J}_{sb}(\tilde{A}_{s,\bullet})] \mathbb{E}[\tilde{A}_{t,s} | \mathcal{J}_{tc}(\tilde{A}_{t,\bullet})]}{m_{s,t}} + O \left(N^2 p \frac{\sqrt{pN}^{k-1}}{\sqrt{N}} \right).$$

Plugging in the estimate for n_{sb} and n_{tc} , and the given estimate for $m_{s,t}$ implies the desired result. $\square$

The following result allows us to analyze the part size evolution in the edge case where $\frac{1}{1-\theta} \in \mathbb{Z}$.

Theorem B.14. *Fix a real parameter $\theta \in (1/2, 1)$ such that $k = \frac{1}{1-\theta} \in \mathbb{Z}$, and constants $T > 1, \delta > 0$ and $\mathcal{T} > 1$. There exists constants $T' > 1, \delta' > 0, \zeta_{k,\mathcal{T}} > 0$ (where $\zeta_{k,\mathcal{T}}$ depends only on $k, \mathcal{T}$) such that for all sufficiently large $N \in \mathbb{Z}$ and $p \in (T^{-1}N^{-\theta}, TN^{-\theta})$ and $\tau \in [\mathcal{T}^{-1}, \mathcal{T}]$, the following statement is true.*

For all $n \in \mathbb{Z}^{\{0,1\}^k}$ and $m \in \mathbb{R}^{\{0,1\}^k \times \{0,1\}^k}$ satisfying

$$\left| n_s - \tilde{n}_s(N) - \tau \frac{N}{\sqrt{pN}} \varepsilon_s \right| \leq T \frac{N}{\sqrt{pN}} N^{-\delta}$$

and

$$\left| m_{s,t} - \tilde{m}_{s,t}(N) \left[1 + \frac{\tau}{\sqrt{pN}} \left(\frac{\varepsilon_s}{\nu_s} + \frac{\varepsilon_t}{\nu_t} \right) \right] \right| \leq T \frac{1}{\sqrt{pN}} N^{-\delta} N^2 p,$$

there exists $q \in (0,1)^{\{0,1\}^k \times \{0,1\}^k}$ satisfying

$$\left| q_{s,t} - \frac{\frac{p}{1-p} e^{\frac{\gamma_{s,t}}{\nu_t \sqrt{pN}}}}{1 + \frac{p}{1-p} e^{\frac{\gamma_{s,t}}{\nu_t \sqrt{pN}}}} \right| \leq T \frac{p}{\sqrt{pN}} N^{-\delta'}$$

such that

$$n_s \mathbb{E}[A_{s,t} | \mathcal{I}_s(A_{s,\bullet})] = m_{s,t}$$

where $A_{s,t} \sim \text{Bin}(n_s - \mathbf{1}_{s=t}, q_{s,t})$ are all sampled independently, and for all $s, t \in \{0,1\}^k$ and $b, c \in \{0,1\}$, we have

$$n_{sb} \geq_{\bar{b}b} N \left(\nu_{sb} + (-1)^b \zeta_{k,\mathcal{T}} \right)$$

where

$$n_{sb} = n_s \mathbb{P}[\mathcal{J}_{sb}(A_{s,\bullet}) | \mathcal{I}_s(A_{s,\bullet})].$$

Proof. This proof closely follows that of [Theorem B.11](#), with most arguments copied with very minor modifications.

For all $s \in \{0,1\}^k$ let

$$\mathcal{D}_s := \left\{ x \in \mathbb{R}^{\{0,1\}^k} : \sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} x_t \geq_{\bar{s}_r s_r} 0 \quad \forall 1 \leq r \leq k-1 \right\}$$

and define the functions

$$f_{s,N} : \mathbb{R}^{\{0,1\}^k} \rightarrow \mathcal{D}_s$$

$$\sigma \mapsto \left(\frac{\mathbb{E}[A_t | \mathcal{I}_s(A)] - p \left(\tilde{n}_t(N) + \tau \frac{N}{\sqrt{pN}} \varepsilon_t \right)}{\sqrt{pN}} \right)_{t \in \{0,1\}^k}$$

where we sample $A_t \sim \text{Bin} \left(n_t, \frac{\frac{p}{1-p} e^{\frac{\sigma_t}{\nu_t \sqrt{pN}}}}{1 + \frac{p}{1-p} e^{\frac{\sigma_t}{\nu_t \sqrt{pN}}}} \right)$ for all $t \in \{0,1\}^k$ all independently, and

$$f_s : \mathbb{R}^{\{0,1\}^k} \rightarrow \mathcal{D}_s$$

$$\sigma \mapsto \mathbb{E}[W | \mathcal{I}_s(W)]$$

where we sample $W_t \sim \mathcal{N}(\sigma_t, \nu_t)$ for all $t \in \{0,1\}^k$ all independently. Note that $f_{s,N}$ is well defined due to [Lemma B.9](#). By [Theorem A.2](#), there exists $\delta' > 0$ such that for all $T' > 0$, there exists $C(T') > 0$ independent of N such that

$$|f_{s,N}(\sigma) - f_s(\sigma)| \leq N^{-\delta'} C(T')$$

for all $|\sigma| < T'$ and all sufficiently large N . Furthermore, [Theorem B.2](#) implies that f_s is a strong bijection.

Let $y^{(s)} \in \mathbb{R}^{\{0,1\}^k}$ be defined by

$$y_t^{(s)} = \frac{\frac{m_{s,t}}{n_s} - p \left(\tilde{n}_t(N) + \tau \frac{N}{\sqrt{pN}} \varepsilon_s \right)}{\sqrt{pN}}.$$

We find that $y_t^{(s)} = \nu_t \mu_{s,t} + O(N^{-\delta'})$ (the ε_s and ε_t terms cancel out), which by the proof of [Definition B.7](#) implies that there is a constant T' such that $\|y^{(s)}\|_\infty < T'$ and $y^{(s)} \in \mathcal{D}_s$ and $\left\| \sum_{t \in \{0,1\}^k} (-1)^{t_{r-1}} y_t^{(s)} \right\|_\infty \geq T'^{-1}$ for all $1 \leq r \leq k-1$. Thus, by [Theorem B.6](#), there is a constant $C > 0$ such that there exists $\sigma_s \in \mathbb{R}^{\{0,1\}^k}$ with $f_{s,N}(\sigma_s) = y^{(s)}$ and $\|\sigma_s - f_s^{-1}(y^{(s)})\|_\infty \leq CN^{-\delta'}$.

Now select

$$q_{s,t} = \frac{\frac{p}{1-p} e^{\frac{(\sigma_s)_t}{\nu_t \sqrt{pN}}}}{1 + \frac{p}{1-p} e^{\frac{(\sigma_s)_t}{\nu_t \sqrt{pN}}}}.$$

Note that since $\|\gamma_{s,\bullet} - f_s^{-1}(y^{(s)})\|_\infty = O(N^{-\delta'})$ (as $\gamma_{s,\bullet} = f_s^{-1}(\nu_\bullet \mu_{s,\bullet})$, and f^{-1} has bounded gradient in a compact set), so $\|\sigma_s - \gamma_{s,\bullet}\|_\infty = O(N^{-\delta'})$, so

$$q_{s,t} = \frac{\frac{p}{1-p} e^{\frac{\gamma_{s,t}}{\nu_t \sqrt{pN}}}}{1 + \frac{p}{1-p} e^{\frac{\gamma_{s,t}}{\nu_t \sqrt{pN}}}} + O\left(\frac{p}{\sqrt{pN}} N^{-\delta'}\right).$$

By [Theorem A.2](#), we have that there is some constant $\delta'' > 0$ such that

$$\begin{aligned} \mathbb{P}[\mathcal{J}_{sb}(A_{s,\bullet}) | \mathcal{I}_s(A_{s,\bullet})] &= \mathbb{P}\left[\mathcal{J}_{sb}\left(p\left(\tilde{n}_\bullet(N) + \tau \frac{N}{\sqrt{pN}} \varepsilon_\bullet\right) + \sqrt{pN} W_{s,\bullet}\right) \middle| \mathcal{I}_s(W_{s,\bullet})\right] + O(N^{-\delta''}) \\ &= \mathbb{P}[\mathcal{J}_{sb}(W_{s,\bullet} + \tau \varepsilon_\bullet) | \mathcal{I}_s(W_{s,\bullet})] + O(N^{-\delta''}) \\ &= \mathbb{P}\left[\sum_{t \in \{0,1\}^k} (-1)^{t_{k-1}} W_{s,t} + \tau \sum_{t \in \{0,1\}^k} (-1)^{t_{k-1}} \varepsilon_t \geq_{\bar{b}b} 0 \middle| \mathcal{I}_s(W_{s,\bullet})\right] + O(N^{-\delta''}), \end{aligned}$$

where recall that $W_{s,t} \sim \mathcal{N}(\gamma_{s,t}, \nu_t)$ are all sampled independently. By [Lemma B.10](#) we have $\sum_{t \in \{0,1\}^k} (-1)^{t_{k-1}} \varepsilon_t > 0$, so there exists some constant $\zeta'_{k,\mathcal{T}} > 0$ such that

$$\begin{aligned} &\mathbb{P}\left[\sum_{t \in \{0,1\}^k} (-1)^{t_{k-1}} W_{s,t} + \tau \sum_{t \in \{0,1\}^k} (-1)^{t_{k-1}} \varepsilon_t \geq_{\bar{b}b} 0 \middle| \mathcal{I}_s(W_{s,\bullet})\right] \\ &\geq_{\bar{b}b} \mathbb{P}\left[\sum_{t \in \{0,1\}^k} (-1)^{t_{k-1}} W_{s,t} \geq_{\bar{b}b} 0 \middle| \mathcal{I}_s(W_{s,\bullet})\right] + (-1)^b \zeta'_{k,\mathcal{T}} \\ &= \frac{\nu_{sb}}{\nu_s} + (-1)^b \zeta'_{k,\mathcal{T}} \end{aligned}$$

for all $s \in \{0,1\}^k$, which directly implies the result. $\square$

APPENDIX C. GRAPH ENUMERATION RESULTS

C.1. Transference. We have the following definitions of degree sequences in graphs and bigraphs respectively.

Definition C.1. Let $\mathcal{D}_{n,m}^{\text{graph}}$ be the set of all sequences $\mathbf{d} \in [0, n-1]^n$ such that $\sum_i d_i = 2m$. Let $\mathbb{P}_{G(n,m)}$ be the probability distribution on $\mathcal{D}_{n,m}^{\text{graph}}$ given by sampling the degrees of the random graph $G(n, m)$. Let $\mathbb{P}_{B(n,m)}$ be the probability distribution on $\mathcal{D}_{n,m}^{\text{graph}}$ given by sampling $d_i \sim \text{Bin}(n-1, \frac{1}{2})$ and conditioning on $\sum_i d_i = 2m$.

Definition C.2. Let $\mathcal{D}_{\ell,n,m}^{\text{bigraph}}$ be the set of all sequences $\mathbf{d} = (\mathbf{s}, \mathbf{t}) \in [0, n]^\ell \times [0, \ell]^n$ such that $\sum_i s_i = \sum_j t_j = m$. Let $\mathbb{P}_{G(\ell,n,m)}$ be the probability distribution on $\mathcal{D}_{\ell,n,m}^{\text{bigraph}}$ given by sampling the degrees of the random bigraph $G(\ell, n, m)$. Let $\mathbb{P}_{B(\ell,n,m)}$ be the probability distribution on $\mathcal{D}_{\ell,n,m}^{\text{bigraph}}$ given by sampling $s_i \sim \text{Bin}(n, \frac{1}{2})$ and $t_j \sim \text{Bin}(\ell, \frac{1}{2})$, and conditioning on $\sum_i s_i = \sum_j t_j = m$.

We have the following imported results.

Theorem C.3 ([1] Theorem 1.4). *Let $\mu_0 > 0$ be a sufficiently small constant, and let $1/2 < \alpha < 3/5$. Let n and m be fixed integers, and let $d = 2m/n$. Assume that $\mu = \frac{d}{n-1}$ satisfies $\mu \leq \mu_0$, and for all fixed $K > 0$, $\log^K n/n = o(\mu)$. Uniformly over all $\mathbf{d} \in \mathcal{D}_{n,m}^{\text{graph}}$ with $|d_i - d| \leq d^\alpha$ for all i , we have*

$$\mathbb{P}_{G(n,m)}(\mathbf{d}) = \mathbb{P}_{B(n,m)}(\mathbf{d}) \exp\left(\frac{1}{4} - \frac{\gamma_2(\mathbf{d})^2}{4\mu^2(1-\mu)^2}\right) \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}} + d^{5\alpha-3}\right)\right),$$

where $\gamma_2(\mathbf{d}) = \frac{1}{(n-1)^2} \sum_i (d_i - d)^2$.

Theorem C.4 ([2] Theorem 1.1). *Let $\mu_0 > 0$ be a sufficiently small constant, and let $1/2 < \alpha < 3/5$. Let ℓ, n, m be fixed integers such that*

$$\mu := \frac{m}{n\ell} < \mu_0 \text{ and } (\ell + n)^{5-5\alpha} = o(\ell n m^{3-5\alpha}),$$

and for all fixed $K > 0$, $\ell \log^K n + n \log^K \ell = o(m)$. Let $s = m/\ell$ and $t = n/\ell$.

Uniformly over all $(\mathbf{s}, \mathbf{t}) \in \mathcal{D}_{\ell,n,m}^{\text{bigraph}}$ with $|s_i - s| \leq s^\alpha$ for all i and $|t_j - t| \leq t^\alpha$ for all j , we have

$$\mathbb{P}_{G(\ell,n,m)}((\mathbf{s}, \mathbf{t})) = \mathbb{P}_{B(\ell,n,m)}((\mathbf{s}, \mathbf{t})) H(\mathbf{s}, \mathbf{t}) \left(1 + O\left(\frac{\log^2 \ell}{\sqrt{\ell}} + \frac{\log^2 n}{\sqrt{n}} + \min\{s, t\}^{5\alpha-5} \frac{m^2}{\ell n}\right)\right),$$

where

$$H(\mathbf{s}, \mathbf{t}) = \exp\left(-\frac{1}{2} \left(1 - \frac{\sigma^2(\mathbf{s})}{s(1-\mu)}\right) \left(1 - \frac{\sigma^2(\mathbf{t})}{t(1-\mu)}\right)\right)$$

where $\sigma^2(\mathbf{s}) = \frac{1}{\ell} \sum_i (s_i - s)^2$ and $\sigma^2(\mathbf{t}) = \frac{1}{n} \sum_j (t_j - t)^2$.

C.2. Second Moment Bound. We have the following lemma that is analogous to Lemma 5.2 of [4].

Lemma C.5. *Fix a real parameter $\theta \in (1/2, 1)$ and constants $T > 1$ and $K > 0$. There exists a constant $C > 0$ such that for all sufficiently large $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^{-\theta})$, the following are true.*

- *Sample $G \sim \mathbb{G}(n, p)$ and let $(d_1, \dots, d_n)$ be the resulting degree sequence, with mean d . Then,*

$$\mathbb{P}\left[\sum_{i=1}^n (d_i - d)^2 \geq Cpn^2\right] \leq \exp(-Kn).$$

- *Consider $\ell \in \mathbb{Z}$ with $\ell \in [T^{-1}n, Tn]$. Sample $G \sim \mathbb{G}(\ell, n, p)$ and let $(s_1, \dots, s_\ell), (t_1, \dots, t_n)$ be the resulting degree sequence, with means s and t respectively. Then,*

$$\mathbb{P}\left[\sum_{i=1}^{\ell} (s_i - s)^2 \geq Cpln \text{ or } \sum_{j=1}^n (t_j - t)^2 \geq Cpln\right] \leq \exp(-Kn).$$

Proof. We only prove the graph case, as the proof in the bigraph case is extremely similar.

Fix $C > 0$. Let $\mathcal{E}$ be the event that $\sum_{i=1}^n (d_i - pn)^2 \geq Cpn^2$.

Indeed, suppose that

$$\sum_{i=1}^n (d_i - pn)^2 \geq Cpn^2.$$

Let T be a uniformly random set subset of $[n]$, and let $U = [n] \setminus T$. Let $\mathbb{E}_T$ denote the result of averaging only over the randomness of T . Note that for fixed $v \in [n]$, $\mathbf{1}_{v \in T}$ and $\deg(v, T)$ are independent random variables, so we have

$$\begin{aligned} & \mathbb{E}_T [\mathbf{1}_{v \in U} (\deg(v, T) - p|T|)^2 + \mathbf{1}_{v \in T} (\deg(v, U) - p|U|)^2] \\ &= \frac{1}{2} \mathbb{E}_T [(\deg(v, T) - p|T|)^2 + (\deg(v, U) - p|U|)^2] \\ &\geq \frac{1}{4} \mathbb{E}_T [(\deg v - pn)^2] \\ &= \frac{1}{4} (d_v - pn)^2. \end{aligned}$$

Summing over all v , we have

$$\mathbb{E}_T \left[\sum_{v \in [n]} (\mathbf{1}_{v \in U} (\deg(v, T) - p|T|)^2 + \mathbf{1}_{v \in T} (\deg(v, U) - p|U|)^2) \right] \geq Cpn^2/4,$$

so there is some $T \subset [n]$ with

$$\sum_{v \in [n] \setminus T} (\deg(v, T) - p|T|)^2 \geq Cpn^2/8.$$

Let $\mathcal{E}_T$ denote the event where the above inequality holds. Thus, we see that if $\mathcal{E}$ holds, then $\mathcal{E}_T$ holds for some T , so we have

$$\mathbb{P}[\mathcal{E}] \leq \sum_{T \subset [n]} \mathbb{P}[\mathcal{E}_T].$$

We now upper bound $\mathbb{P}[\mathcal{E}_T]$. The main tool is the Bernstein inequality as stated in Theorem 5.1 of [4].

Indeed, we see that over the randomness of G , $(\deg(v, T) - p|T|)^2$ is distributed as X^2 , where $X \sim \text{Bin}(|T|, p) - p|T|$. Furthermore, all the $(\deg(v, T) - p|T|)^2$ are independent as we vary $v \in [n] \setminus T$. It is easy to check that $\|X^2\|_{\psi_1} \leq 100p|T|$, so by the Bernstein inequality, there is an absolute constant c such that

$$\mathbb{P} \left[\sum_{v \in [n] \setminus T} (\deg(v, T) - p|T|)^2 \geq t \right] \leq 2 \exp \left(-c \min \left\{ \frac{t^2}{(n - |T|) \cdot (100p|T|)^2}, \frac{t}{100p|T|} \right\} \right).$$

Thus, there is some other absolute constant c' such that

$$\mathbb{P} \left[\sum_{v \in [n] \setminus T} (\deg(v, T) - p|T|)^2 \geq Cpn^2/8 \right] \leq \exp(-c' Cn).$$

Putting everything together, we have that

$$\mathbb{P}[\mathcal{E}] \leq 2^n \exp(-c' Cn).$$

Choosing C sufficiently large compared to K finishes the proof. $\square$

We have the following basic counting lemma that we use in our second moment bound analysis.

Lemma C.6. Fix a real parameter $\theta \in (1/2, 1)$ and constants $T > 1$ and $C > 0$. There exists a constant $C' > 0$ such that for all $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^{-\theta})$, the number of integer sequences

$$|\{(y_1, \dots, y_n) \in ([-C\sqrt{np}, C\sqrt{np}] \cap \mathbb{Z})^n : y_1 + \dots + y_n = 0\}|$$

is at least $\exp(-C'n)(\sqrt{np})^n$.

Proof. Note that at least half of all sequences

$$(y_1, \dots, y_{n-n^{2/3}}) \in [-C\sqrt{np}, C\sqrt{np}]^{n-n^{2/3}}$$

have sum bounded above by some constant times $\sqrt{n}\sqrt{np}$ by Chebyshev, and there is clearly at least one continuation of the rest of the $n^{2/3}$ elements of the sequence that make the overall sum 0. Therefore, the number of such sequences is at least

$$\frac{1}{2}(\sqrt{np})^{n-n^{2/3}} \geq \exp(-C'n)(\sqrt{np})^n,$$

for some constant $C' > 0$, as desired. $\square$

C.3. Edge Enumeration. We have the following result, which is a corollary of Theorem 1.6 of [1] and Theorem 1.5 of [2].

Lemma C.7. Fix a real parameter $\theta \in (1/2, 1)$ and a constant $T > 1$. For all sufficiently large $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^{-\theta})$, the following are true.

- Select $m \in \mathbb{Z}$ satisfying $|m - pn(n-1)/2| \leq Tn^2p/\sqrt{pn}$, and select $\mathbf{d} \in \mathcal{D}_{n,m}^{\text{graph}}$ such that $|d_i - pn| \leq (pn)^{4/7}$ for all $i \in [n]$. Pick a subset $U \subset [n]$ satisfying $|U|, n - |U| \geq T^{-1}n$. Sample G uniformly at random from all graphs with degree sequence $\mathbf{d}$, and let $E = |E(G[U])|$ (the number of edges in the induced subgraph on vertex set U). Then,

$$\mathbb{P} \left[\left| E - \frac{(\sum_{x \in U} d_x)^2}{4m} \right| \geq n^2 p \sqrt{\frac{(pn)^{1/7}}{n} \log n} \right] < \frac{1}{\log n}.$$

- Select $m \in \mathbb{Z}$ satisfying $|m - pn(n-1)/2| \leq Tn^2p/\sqrt{pn}$, and select $\mathbf{d} \in \mathcal{D}_{n,m}^{\text{graph}}$ such that $|d_i - pn| \leq (pn)^{4/7}$ for all $i \in [n]$. Pick a subset $U \subset [n]$ satisfying $|U|, n - |U| \geq T^{-1}n$. Sample G uniformly at random from all graphs with degree sequence $\mathbf{d}$, and let $E = |E(G[U, [n] \setminus U])|$ (the number of edges in the induced bipartite subgraph with vertex sets U and $[n] \setminus U$). Then,

$$\mathbb{P} \left[\left| E - \frac{(\sum_{x \in U} d_x) \left(\sum_{y \in [n] \setminus U} d_y \right)}{2m} \right| \geq n^2 p \sqrt{\frac{(pn)^{1/7}}{n} \log n} \right] < \frac{1}{\log n}.$$

- Select $\ell \in \mathbb{Z} \cap [T^{-1}n, Tn]$. Select $m \in \mathbb{Z}$ satisfying $|m - p\ell n| \leq Tn^2p/\sqrt{pn}$, and select $(\mathbf{s}, \mathbf{t}) \in \mathcal{D}_{\ell,n,m}^{\text{bigraph}}$ such that $|s_i - p\ell| \leq (pn)^{4/7}$ for all $i \in [\ell]$ and $|t_j - p\ell| \leq (pn)^{4/7}$ for all $j \in [n]$. Pick subsets $U \subset [\ell]$ and $V \subset [n]$ satisfying $|U|, \ell - |U|, |V|, \ell - |V| \geq T^{-1}n$. Sample G uniformly at random from all bigraphs with degree sequence $(\mathbf{s}, \mathbf{t})$, and let $E = |E(G[U, V])|$ (the number of edges in the induced bipartite subgraph with vertex sets U and V). Then,

$$\mathbb{P} \left[\left| E - \frac{(\sum_{x \in U} s_x) \left(\sum_{y \in V} t_y \right)}{m} \right| \geq n^2 p \sqrt{\frac{(pn)^{1/7}}{n} \log n} \right] < \frac{1}{\log n}.$$

Proof. We will use Theorem 1.6 of [1] to prove the first bullet point. The proof of the second bullet point is identical, and the third is identical, where we use Theorem 1.5 of [2] instead.

Note that

$$|E| = \sum_{\{a,b\} \subset U} \mathbf{1}_{ab \in E}.$$

By Theorem 1.6 of [1], we see that

$$\mathbb{E}[|E|] = \left(1 + O\left(\frac{(pn)^{1/7}}{n}\right)\right) \sum_{\{a,b\} \subset U} \frac{d_a d_b}{2m} = \frac{1}{4m} \left(\sum_{a \in U} d_a\right)^2 + O\left((np)^{8/7}\right).$$

In order to compute the variance, we have to compute quantities of the form $\mathbb{P}[ab \in E \text{ and } cd \in E]$. We begin with the case where edges ab and cd do not share any vertices.

Our goal is to compute $\mathbb{P}[cd \in E | ab \in E]$. Reveal the edges in the graph incident to a and b . The rest of the graph is given again uniformly over all graphs with the right degree sequence. Note that the degree sequence for the remaining graph is the same as $\mathbf{d}$ except with some vertices with degree potentially 1 or 2 lower, and a total of two less vertices. It is easy to see that this doesn't effect our estimate for the probability that $cd \in E$, so we in fact have

$$\mathbb{P}[ab \in E \text{ and } cd \in E] = \left(1 + O\left(\frac{(pn)^{1/7}}{n}\right)\right) \frac{d_a d_b d_c d_d}{(2m)^2},$$

if ab and cd are not incident to each other.

Let's now calculate the probability in the case that $d = b$, so we're trying to find the probability that the path abc is in the graph G . The same approach as above works, where our goal is to compute $\mathbb{P}[bc \in E | ab \in E]$. Begin by revealing all edges incident to a , and note the rest of the graph is drawn uniformly given a fixed degree sequence $\mathbf{d}'$ which is given by $\mathbf{d}$ with one entry removed, and some entries decreased by at most 1. This doesn't affect the edge probability of $bc \in E$ up to the error term we care about, so we again have

$$\mathbb{P}[ab \in E \text{ and } bc \in E] = \left(1 + O\left(\frac{(pn)^{1/7}}{n}\right)\right) \frac{d_a d_b^2 d_c}{(2m)^2}.$$

Thus, we see that

$$\begin{aligned} \mathbb{E}[|E|^2] &= \left(1 + O\left(\frac{(pn)^{1/7}}{n}\right)\right) \left[\left(\sum_{\{a,b\} \subset U} \frac{d_a d_b}{2m}\right)^2 + \sum_{\{a,b\} \subset U} \left(\frac{d_a d_b}{2m} - \frac{d_a^2 d_b^2}{(2m)^2}\right) \right] \\ &= \mathbb{E}[|E|]^2 + O(n^3 p^2 (pn)^{1/7}), \end{aligned}$$

so

$$\sqrt{\text{Var}[|E|]} = O\left(n^2 p \sqrt{\frac{(pn)^{1/7}}{n}}\right).$$

The result now follows by Chebyshev's inequality. $\square$

C.4. Degree Regularity Results. We prove results about the degrees to vertex subsets in random graphs using the graph enumeration results from above. The calculations mostly follow a simplified version of those in Appendix A of [3].

Our key result is as follows.

Theorem C.8. *Fix a real parameter $\theta \in (1/2, 1)$ and a constant $T > 1$. There exists a constant $C > 0$ such that for all sufficiently large $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^{-\theta})$, the following are true.*

- *Select $m \in \mathbb{Z}$ satisfying $|m - pn(n-1)/2| \leq Tn^2 p / \sqrt{pn}$, and select $\mathbf{d} \in \mathcal{D}_{n,m}^{\text{graph}}$ such that $d_i = pn + \beta_i \sqrt{pn}$ where $|\beta_i| \leq \log n$. Pick a vertex $v \in [n]$ and a subset $S \subset [n]$ with size $h = |S|$ satisfying $h, n-h \geq T^{-1}n$. Pick an integer $t \in [0, d_v]$ and write $t = ph + \tau \sqrt{ph}$.*

Sample G uniformly at random from all graphs with degree sequence $\mathbf{d}$. If $|\tau| \geq \log^{100} n$, then

$$\mathbb{P}[\deg_S(v) = t] \leq \exp(-C\tau^2).$$

- Select $\ell \in \mathbb{Z} \cap [T^{-1}n, Tn]$. Select $m \in \mathbb{Z}$ satisfying $|m - p\ell n| \leq Tn^2p/\sqrt{pn}$, and select $(\mathbf{s}, \mathbf{t}) \in \mathcal{D}_{\ell, n, m}^{\text{bigraph}}$ such that $s_i = pn + \alpha_i\sqrt{pn}$ and $t_j = p\ell + \beta_j\sqrt{p\ell}$, where $|\alpha_i|, |\beta_j| \leq \log n$ for all $i \in [\ell]$ and $j \in [n]$. Pick a vertex $v \in [\ell]$ and a subset $S \subset [n]$ with size $h = |S|$ satisfying $h, n - h \geq T^{-1}n$. Pick an integer $t \in [0, s_v]$ and write $t = ph + \tau\sqrt{ph}$. Sample G uniformly at random from all bigraphs with degree sequence $(\mathbf{s}, \mathbf{t})$. If $|\tau| \geq \log^{100} n$, then

$$\mathbb{P}[\deg_S(v) = t] \leq \exp(-C\tau^2).$$

We prove this through a series of lemmas. We begin with a pair of technical estimates.

Lemma C.9. Fix a real parameter $\theta \in (1/2, 1)$ and a constant $T > 1$. For all sufficiently large $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^{-\theta})$, the following are true.

- Select $m \in \mathbb{Z}$ satisfying $|m - pn(n-1)/2| \leq Tn^2p/\sqrt{pn}$, and select $d_v \in \mathbb{Z}$ satisfying $|d_v - pn| \leq \sqrt{pn} \log n$. Then,

$$\frac{\binom{(n-1)(n-2)/2}{m-d_v} \binom{(n-1)(n-2)}{2m-2d_v}^{-1}}{\binom{n(n-1)/2}{m} \binom{n(n-1)}{2m}^{-1}} = \exp(O(\log n)) p^{-d_v} e^{pn}.$$

- Select $\ell \in \mathbb{Z} \cap [T^{-1}n, Tn]$. Select $m \in \mathbb{Z}$ satisfying $|m - p\ell n| \leq Tn^2p/\sqrt{pn}$, and select $s_v \in \mathbb{Z}$ satisfying $|s_v - pn| \leq \sqrt{pn} \log n$. Then,

$$\frac{\binom{\ell n}{m}}{\binom{\ell(n-1)}{m-s_v}} = \exp(O(\log n)) p^{-s_v} e^{pn}.$$

Proof. We prove the first result, as the second one has a similar proof.

Note that

$$\frac{\binom{(n-1)(n-2)/2}{m-d_v}}{\binom{n(n-1)/2}{m}} = \left[\frac{\left[\frac{n(n-1)}{2} \right]!}{\left[\frac{n(n-1)}{2} - (n-1) \right]!} \right]^{-1} \left[\frac{m!}{(m-d_v)!} \right] \left[\frac{\left[\frac{n(n-1)}{2} - m \right]!}{\left[\frac{n(n-1)}{2} - m - (n-1-d_v) \right]!} \right].$$

We find that

$$\begin{aligned} \log \left[\frac{\left[\frac{n(n-1)}{2} \right]!}{\left[\frac{n(n-1)}{2} - (n-1) \right]!} \right] &= \sum_{i=0}^{n-2} \log \left(\frac{n(n-1)}{2} - i \right) \\ &= (n-1) \log \left(\frac{n(n-1)}{2} \right) + \sum_{i=0}^{n-2} \log \left(1 - \frac{i}{\frac{n(n-1)}{2}} \right) \\ &= (n-1) \log \left(\frac{n(n-1)}{2} \right) - \sum_{i=0}^{n-2} \frac{i}{\frac{n(n-1)}{2}} + O(1/n) \\ &= (n-1) \log \left(\frac{n(n-1)}{2} \right) - 1 + O(1/n) \end{aligned}$$

and

$$\begin{aligned} \log \left[\frac{m!}{(m-d_v)!} \right] &= d_v \log m + \sum_{i=0}^{d_v-1} \log \left(1 - \frac{i}{m} \right) \\ &= d_v \log m + O(d_v^2/m) \end{aligned}$$

$$= d_v \log m + O(p)$$

and

$$\begin{aligned} \log \left[\frac{\left[\frac{n(n-1)}{2} - m \right]!}{\left[\frac{n(n-1)}{2} - m - (n-1-d_v) \right]!} \right] &= \sum_{i=0}^{n-2-d_v} \log \left(\frac{n(n-1)}{2} - m - i \right) \\ &= (n-1-d_v) \log \left(\frac{n(n-1)}{2} - m \right) + \sum_{i=0}^{n-2-d_v} \log \left(1 - \frac{i}{\frac{n(n-1)}{2} - m} \right) \\ &= (n-1-d_v) \log \left(\frac{n(n-1)}{2} - m \right) - \sum_{i=0}^{n-2-d_v} \frac{i}{\frac{n(n-1)}{2} - m} + O(1/n) \\ &= (n-1-d_v) \log \left(\frac{n(n-1)}{2} - m \right) - 1 + O(p), \end{aligned}$$

so

$$\frac{\binom{(n-1)(n-2)/2}{m-d_v}}{\binom{(n-1)(n-2)/2}{m}} = (1 + O(p)) \left(\frac{n(n-1)}{2} \right)^{1-n} m^{d_v} \left(\frac{n(n-1)}{2} - m \right)^{n-1-d_v}.$$

We similarly derive

$$\frac{\binom{(n-1)(n-2)}{2m-2d_v}}{\binom{(n-1)(n-2)}{2m}} = (1 + O(p)) (n(n-1))^{3-2n} (2m)^{2d_v} (n(n-1) - 2m)^{2n-3-2d_v}.$$

Thus,

$$\frac{\binom{(n-1)(n-2)/2}{m-d_v} \binom{(n-1)(n-2)}{2m-2d_v}^{-1}}{\binom{(n-1)(n-2)/2}{m} \binom{(n-1)(n-2)}{2m}^{-1}} = (1 + O(p)) \left[\frac{n(n-1)}{2m} \right]^{d_v} \left[\frac{\frac{n(n-1)}{2}}{\frac{n(n-1)}{2} - m} \right]^{n-2-d_v}.$$

We see that

$$\begin{aligned} \log \left(\left[\frac{n(n-1)}{2m} \right]^{d_v} \right) &= d_v \log \left(\frac{n^2}{2m} \right) + O(p) \\ &= -d_v \log p - d_v \log \left(\frac{2m}{pn^2} \right) + O(p) \\ &= -d_v \log p - d_v \left[\frac{2m}{pn^2} - 1 \right] + O(1/pn) \\ &= -d_v \log p - \frac{2m}{n} + pn + O(1/\sqrt{pn}) \end{aligned}$$

and

$$\begin{aligned} \log \left(\left[\frac{\frac{n(n-1)}{2}}{\frac{n(n-1)}{2} - m} \right]^{n-2-d_v} \right) &= (n-2-d_v) \log \left[\frac{1}{1 - \frac{2m}{n(n-1)}} \right] \\ &= \frac{2m}{n} + O(np^2). \end{aligned}$$

Adding these two directly implies the desired result. $\square$

Lemma C.10. *Fix a real parameter $\theta \in (1/2, 1)$ and a constant $\mathcal{T} > 1$. For all sufficiently large $n \in \mathbb{Z}$ and $p \in (\mathcal{T}^{-1}n^{-\theta}, \mathcal{T}n^{-\theta})$, the following are true.*

- Select $m \in \mathbb{Z}$ satisfying $|m - pn(n-1)/2| \leq \mathcal{T}n^2p/\sqrt{pn}$, and select $\mathbf{d} \in \mathcal{D}_{n,m}^{\text{graph}}$ such that $d_i = pn + \beta_i\sqrt{pn}$ where $|\beta_i| \leq \log n$. Pick a vertex $v \in [n]$ and a subset $S \subset [n]$ with size $h = |S|$ satisfying $h, n-h \geq \mathcal{T}^{-1}n$. Pick an integer $t \in [0, d_v]$. Sample G uniformly at random from all graphs with degree sequence $\mathbf{d}$. We have

$$\mathbb{P}[\deg_S(v) = t] = \exp(O(\log^4 n)) \frac{\binom{h-1_S(v)}{t} \binom{n-h-1_{[n]\setminus S}(v)}{d_v-t}}{\binom{n-1}{d_v}} \mathbb{E}_T[\exp(\Lambda_T)]$$

where $T = T_1 \cup T_2$ is a random set chosen by picking T_1 uniformly from $\binom{S \setminus v}{t}$ and T_2 uniformly from $\binom{[n] \setminus S \cup v}{d_v-t}$, and where

$$\Lambda_T := \sum_{i \in T} \frac{\beta_i}{\sqrt{pn}} - \sum_{i \in [n] \setminus v \cup T} \frac{\sqrt{pn}}{n} \beta_i.$$

- Select $\ell \in \mathbb{Z} \cap [\mathcal{T}^{-1}n, \mathcal{T}n]$. Select $m \in \mathbb{Z}$ satisfying $|m - p\ell n| \leq \mathcal{T}n^2p/\sqrt{pn}$, and select $(\mathbf{s}, \mathbf{t}) \in \mathcal{D}_{\ell, n, m}^{\text{bigraph}}$ such that $s_i = pn + \alpha_i\sqrt{pn}$ and $t_j = p\ell + \beta_j\sqrt{p\ell}$, where $|\alpha_i|, |\beta_j| \leq \log n$ for all $i \in [\ell]$ and $j \in [n]$. Pick a vertex $v \in [\ell]$ and a subset $S \subset [n]$ with size $h = |S|$ satisfying $h, n-h \geq \mathcal{T}^{-1}n$. Pick an integer $t \in [0, s_v]$. Sample G uniformly at random from all bigraphs with degree sequence $(\mathbf{s}, \mathbf{t})$. We have

$$\mathbb{P}[\deg_S(v) = t] = \exp(O(\log^4 n)) \frac{\binom{h}{t} \binom{n-h}{s_v-t}}{\binom{n}{s_v}} \mathbb{E}_T[\exp(\Lambda_T)]$$

where $T = T_1 \cup T_2$ is a random set chosen by picking T_1 uniformly from $\binom{S}{t}$ and T_2 uniformly from $\binom{[n] \setminus S}{s_v-t}$, and where

$$\Lambda_T := \sum_{j \in T} \frac{\beta_j}{\sqrt{p\ell}} - \sum_{j \in [n] \setminus T} \frac{\sqrt{p\ell}}{\ell} \beta_j.$$

Proof. We prove only the first result, the second has a similar proof but uses [Theorem C.4](#) instead of [Theorem C.3](#), and the second bullet point of [Lemma C.9](#) instead of the first.

Let $\mathbb{P}_{\mathbf{d}}$ denote the uniform distribution on graphs with degree sequence $\mathbf{d}$. Given $T \subset [n] \setminus v$ of size d_v , we begin by computing $\mathbb{P}_{G \sim \mathbb{P}_{\mathbf{d}}}[N_G(v) = T]$. To this end, define

$$\mathbf{d}_T = (d_w - \mathbf{1}_{w \in T})_{w \in [n] \setminus v}.$$

Note that $2m = \sum_{i \in [n]} d_i$ and let $m_T = m - d_v$, and let $d = 2m/n$ and $d_T = 2m_T/(n-1)$. Note that by Bayes' theorem,

$$\mathbb{P}_{G \sim \mathbb{P}_{\mathbf{d}}}[N_G(v) = T] = \mathbb{P}_{G \sim \mathbb{G}(n, m)}[N_G(v) = T] \frac{\mathbb{P}_{\mathbb{G}(n-1, m_T)}[\mathbf{d}_T]}{\mathbb{P}_{\mathbb{G}(n, m)}[\mathbf{d}]} = \frac{\binom{(n-1)(n-2)/2}{m-d_v}}{\binom{(n(n-1)/2)}{m}} \frac{\mathbb{P}_{\mathbb{G}(n-1, m_T)}[\mathbf{d}_T]}{\mathbb{P}_{\mathbb{G}(n, m)}[\mathbf{d}]}.$$

We compute the remaining two probabilities by appealing to [Theorem C.3](#). Indeed, note that $\mu = \frac{d}{n-1} = p + O(p/\sqrt{pn})$, and $\gamma_2(\mathbf{d}) = O(p \log^2 n)$, and similarly $\mu_T = \frac{d_T}{n-2} = p + O(p/\sqrt{pn})$ and $\gamma_2(\mathbf{d}_T) = O(p \log^2 n)$. Thus, by [Theorem C.3](#), we have

$$\mathbb{P}_{G \sim \mathbb{P}_{\mathbf{d}}}[N_G(v) = T] = \exp(O(\log^4 n)) \frac{\binom{(n-1)(n-2)/2}{m-d_v}}{\binom{(n(n-1)/2)}{m}} \frac{\prod_{w \in [n] \setminus v} \binom{n-2}{d_w - \mathbf{1}_{w \in T}}}{\binom{(n-1)(n-2)}{2m-2d_v}} \frac{\prod_{w \in [n]} \binom{n-1}{d_w}}{\binom{(n(n-1))}{2m}}$$

$$= \exp(O(\log^4 n)) \frac{\binom{(n-1)(n-2)/2}{m-d_v} \binom{(n-1)(n-2)}{2m-2d_v}^{-1} \prod_{w \in [n] \setminus v} \binom{n-2}{d_w - \mathbf{1}_{w \in T}}}{\binom{n(n-1)/2}{m} \binom{n(n-1)}{2m}^{-1} \prod_{w \in [n]} \binom{n-1}{d_w}}.$$

Note that

$$\begin{aligned} \frac{\prod_{w \in [n] \setminus v} \binom{n-2}{d_w - \mathbf{1}_{w \in T}}}{\prod_{w \in [n]} \binom{n-1}{d_w}} &= \frac{1}{\binom{n-1}{d_v}} \prod_{i \in T} \frac{d_i}{n-1} \prod_{i \in [n] \setminus v \cup T} \frac{n-1-d_i}{n-1} \\ &= \frac{\exp(O(1))}{\binom{n-1}{d_v}} p^{d_v} (1-p)^{n-1-d_v} \prod_{i \in T} \left[1 + \frac{\beta_i}{\sqrt{pn}}\right] \prod_{i \in [n] \setminus v \cup T} \left[1 - \frac{\sqrt{pn}}{n} \beta_i\right] \\ &= \frac{\exp(O(\log^2 n))}{\binom{n-1}{d_v}} p^{d_v} e^{-np} \exp \left[\sum_{i \in T} \frac{\beta_i}{\sqrt{pn}} - \sum_{i \in [n] \setminus v \cup T} \frac{\sqrt{pn}}{n} \beta_i \right], \end{aligned}$$

so by [Lemma C.9](#), we have

$$\mathbb{P}_{G \sim \mathbb{P}_d}[N_G(v) = T] = \exp(O(\log^4 n)) \frac{\exp(\Lambda_T)}{\binom{n-1}{d_v}}.$$

Now pick T_1 uniformly from $\binom{S \setminus v}{t}$ and T_2 uniformly from $\binom{[n] \setminus v \cup S}{d_v - t}$, and let $T = T_1 \cup T_2$. Under this setup, we have

$$\mathbb{P}_{G \sim \mathbb{P}_d}[\deg_S(v) = t] = \binom{h - \mathbf{1}_S(v)}{t} \binom{n - h - \mathbf{1}_{[n] \setminus S}(v)}{d_v - t} \mathbb{E}_T[\mathbb{P}_{G \sim \mathbb{P}_d}[N_G(v) = T]],$$

so

$$\mathbb{P}[\deg_S(v) = t] = \exp(O(\log^4 n)) \frac{\binom{h - \mathbf{1}_S(v)}{t} \binom{n - h - \mathbf{1}_{[n] \setminus S}(v)}{d_v - t}}{\binom{n-1}{d_v}} \mathbb{E}_T[\exp(\Lambda_T)],$$

as desired. $\square$

We have the following simple lemma that allows us to commute the expectation over the subset T with the exponential.

Lemma C.11. *Fix a real parameter $\theta \in (1/2, 1)$ and a constant $T > 1$. For all sufficiently large $n \in \mathbb{Z}$ and $p \in (T^{-1}n^{-\theta}, Tn^{-\theta})$, the following is true.*

Fix real numbers $a_1, \dots, a_n \in \mathbb{R}$, and consider an integer $s \in [T^{-1}pn, Tpn]$. Pick $\xi \in \{0, 1\}^n$ uniformly at random, and then condition on $\sum_i \xi_i = s$. Let $X = \sum_i \xi_i a_i$. Then,

$$\mathbb{E}[e^X] = O(\sqrt{pn})e^{\mathbb{E}[X]}.$$

Proof. The idea is to let $Y = \sum_i \zeta_i a_i$ where ζ_i are all independently sampled from $\text{Ber}(s/n)$. We see that

$$\mathbb{E}[e^Y] \geq \mathbb{P}[\text{Bin}(n, s/n) = s] \mathbb{E}[e^X] \geq \Omega(1/\sqrt{pn}) \mathbb{E}[e^X]$$

and $\mathbb{E}[e^Y] = e^{\mathbb{E}[X]}$, which completes the proof. $\square$

We are now ready to prove our main result.

Proof of Theorem C.8. We only prove the first result, as the second result has a similar proof but uses the second bullet point of [Lemma C.10](#) instead of the first.

Define Λ_T as in [Lemma C.10](#). We begin by computing its expectation over the randomness of T . Indeed, we see that

$$\mathbb{E}_T[\Lambda_T] = \sum_{x \in S \setminus v} \frac{t}{|S \setminus v|} \frac{\beta_x}{\sqrt{pn}} + \sum_{x \in [n] \setminus v \cup T} \frac{d_v - t}{|[n] \setminus S \cup v|} \frac{\beta_x}{\sqrt{pn}}$$

$$\begin{aligned}
& - \sum_{x \in S \setminus v} \left(1 - \frac{t}{|S \setminus v|}\right) \frac{\sqrt{pn}}{n} \beta_x - \sum_{x \in [n] \setminus S \cup v} \left(1 - \frac{d_v - t}{|[n] \setminus S \cup v|}\right) \frac{\sqrt{pn}}{n} \beta_x \\
& = \sum_{x \in S} \left[\frac{t}{h} - p\right] \frac{\beta_x}{\sqrt{pn}} + \sum_{x \in [n] \setminus S} \left[\frac{d_v - t}{b - h} - p\right] \frac{\beta_x}{\sqrt{pn}} + O(\log n) \\
& = O(|\tau| \log n + \log^2 n).
\end{aligned}$$

Thus, by [Lemma C.10](#) and [Lemma C.11](#), we have

$$\begin{aligned}
\mathbb{P}[\deg_S(v) = t] & = \exp(O(|\tau| \log n + \log^2 n)) \frac{\binom{h-1_{S(v)}}{t} \binom{n-h-1_{[n] \setminus S(v)}}{d_v-t}}{\binom{n-1}{d_v}} \\
& = \exp(O(|\tau| \log n + \log^2 n)) \frac{\binom{h}{t} \binom{n-h}{d_v-t}}{\binom{n}{d_v}}.
\end{aligned}$$

We now bound the hypergeometric term. Indeed, we have that

$$\begin{aligned}
\frac{\binom{h}{t} \binom{n-h}{d_v-t}}{\binom{n}{d_v}} & = \frac{\binom{d_v}{t} \binom{n-d_v}{h-t}}{\binom{n}{h}} \\
& = (1 + o(1)) \binom{d_v}{t} \frac{h^t (n-h)^{d_v-t}}{n^{d_v}} \\
& \leq (1 + o(1)) \mathbb{P} \left[\left| \text{Bin}(d_v, \frac{h}{n}) - d_v \frac{h}{n} \right| \geq (|\tau| - \log n) \sqrt{pn} \right] \\
& \leq (1 + o(1)) \exp \left(-\frac{2(|\tau| - \log n)^2}{1 + \beta_v / \sqrt{pn}} \right),
\end{aligned}$$

where the last inequality follows from Hoeffding's inequality. Thus, since $|\tau| \geq \log^{100} n$, there is some constant $C > 0$ such that

$$\mathbb{P}[\deg_S(v) = t] \leq \exp(-C|\tau|^2),$$

as desired. □

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