The Dynamical $SU(N)$ Symmetry of the Quantum Harmonic Oscillator

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We show that the symmetry group of the N-dimensional isotropic quantum harmonic oscillator is given by SU(N). We analyze the representations generated by the action of the energies on the space of states of fixed energy.

I. INTRODUCTION

The quantum harmonic oscillator is a ubiquitous system in physics, as it is one of the first nontrivial quantum systems that is more or less exactly solvable analytically. It is often the starting point for more complex systems, analyzed using techniques like perturbation theory. In particular, it is the basic building block for quantum field theory, where free fields without interaction are modeled by infinite dimensional harmonic oscillators.

We are interested in the isotropic case, where the potential is radially symmetric. In this case, the energy levels of the harmonic oscillator have an unexpectedly high degree of degeneracy. The nature of the degeneracy is closely related to the symmetries of the system, and understanding the underlying symmetries can lead to a more natural explanation of the supposedly unexpected degeneracy. For example, the degeneracy of the energy levels in the hydrogen atom can be explained by the rotational symmetry of the system, which leads to the classification of states by angular momentum.

In this paper, we show that the symmetry group of the N-dimensional isotropic harmonic oscillator is given by SU(N), the group of $N \times N$ unitary matrices with determinant 1. We look at the action of this symmetry to states of fixed energy, which induces a representation of SU(N). We identify SO(N) (the group of rotations in N-dimensional space) as a subgroup in this representation, and elucidate the connection to angular momentum.

Much of this work was first done by Baker [3] and Demkov [5], with other authors [6],[7] alluding to the SU(N) symmetry but not directly analyzing its action on the Hilbert space of oscillator states.

II. REVIEW OF THE HARMONIC OSCILLATOR

The Hamiltonian for the N-dimensional isotropic harmonic oscillator is given by

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{x}^2,$$

where $\mathbf{x}$ and $\mathbf{p}$ are N-component operators. For the rest of the paper, we will use units where $\hbar = m = \omega = 1$.

Introduce the non-Hermitian *ladder operators*

$$\begin{aligned} a_k &= \tfrac{1}{\sqrt{2}}(x_k + ip_k) \\ a_k^\dagger &= \tfrac{1}{\sqrt{2}}(x_k - ip_k). \end{aligned}$$

The Hamiltonian rewrites as

$$\mathcal{H} = \frac{N}{2} + \sum_{k=1}^{N} a_k^\dagger a_k. \tag{1}$$

Using the standard commutation relations $[x_j, p_k] = i\delta_{jk}$, we derive the commutation relations

$$[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0 \tag{2}$$

$$[a_j, a_k^\dagger] = \delta_{jk}. \tag{3}$$

The energy eigenstates are now given by

$$|n_1, \ldots, n_N\rangle = \prod_{k=1}^{N} \left(a_k^\dagger\right)^{n_k} |0\rangle,$$

with energy

$$\mathcal{H}\,|n_1, \ldots, n_N\rangle = \left(\frac{N}{2} + n_1 + \cdots + n_N\right)|n_1, \ldots, n_N\rangle.$$

Thus, we see that the degeneracy of the energy level $\frac{N}{2}+n$ is $\binom{n+N-1}{N-1}$.

III. SYMMETRIES IN QUANTUM MECHANICS AND LIE ALGEBRAS

Symmetries in a quantum mechanics take the form of transformations given by unitary operators $\mathcal{U}$ that acts on operators $\mathcal{O}$ by

$$\mathcal{O} \to \mathcal{U}^\dagger \mathcal{O} \mathcal{U}. \tag{4}$$

It is particularly interesting to consider the case of *infinitesimal transformations*, which are when $\mathcal{U} = e^{-iS\theta}$ for some Hermitian operator S and infinitesimal θ (we say that S *generates* the transformation). In this case, the transformation affects an operator as

$$\mathcal{O} \to \mathcal{O} + i\theta[S, \mathcal{O}] \tag{5}$$

where we have dropped $O(\theta^2)$ terms (this will be implicit throughout the rest of the paper). In this paper, we only consider continuous transformations (i.e. those that can be decomposed into infinitesimal transformations), so we exclude discussion of things like time reversal symmetry.

We say that a transformation given by $\mathcal{U}$ is a symmetry of the Hamiltonian if the Hamiltonian is unchanged under (4). Note that an infinitesimal transformation generated by S is a symmetry of the Hamiltonian if and only if $[S, \mathcal{H}] = 0$.

The set of symmetries form a Lie group $\mathcal{G}$, and the generators of the infinitesimal form of these symmetries form a Lie algebra $\mathfrak{g}$. Note that $\mathfrak{g}$ is clearly a vector space, and if $S, T \in \mathfrak{g}$, then $[S, \mathcal{H}] = [T, \mathcal{H}] = 0$, so $[[S, T], \mathcal{H}] = 0$, so $[S, T] \in \mathfrak{g}$ as well, which shows that $\mathfrak{g}$ is indeed a Lie algebra. Since we are looking at continuous transformations, every element $\mathcal{U} \in \mathcal{G}$ (including non-infinitesimal transformations) can be written as $\mathcal{U} = e^{-iS}$ for some $S \in \mathfrak{g}$. By a remarkable result of Baker, Campbell, and Hausdorff (see for example [8]), we see that the operator $i\log\left(e^{-iS}e^{-iT}\right)$ can be written purely in terms of successive commutators of S and T, which shows that the group structure of $\mathcal{G}$ is fully determined by the commutator structure of $\mathfrak{g}$. When discussing the theory of Lie algebras, we often abstract away the fact that $\mathfrak{g}$ is made up of Hermitian operators acting on some Hilbert space, and instead view $\mathfrak{g}$ as simply a vector space equipped with a bracket operator $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

It is interesting to note that symmetries of a quantum system lead to conservation laws. In particular, if S generates a symmetry of the Hamiltonian, then S is a *conserved quantity* in the following sense. Let $|\psi\rangle$ be a simultaneous eigenstate of $\mathcal{H}$ and S with eigenvalues E and s (i.e. a state of definite energy and S). Then, the time evolved state $e^{-i\mathcal{H}t}|\psi\rangle$ is also an eigenstate of S with the same eigenvalue, since

$$Se^{-i\mathcal{H}t}|\psi\rangle = e^{-i\mathcal{H}t}S|\psi\rangle = se^{-i\mathcal{H}t}|\psi\rangle .$$

This means that states of definite S conserve their value of S throughout time evolution.

Common examples of symmetries leading to conservation laws are spacetime translation symmetry leading to energy/momentum conservation and rotations leading to angular momentum conservation.

For a more thorough review of the theory of symmetries in quantum mechanics, see [2], or any other standard quantum mechanics text.

IV. SYMMETRY IN THE HARMONIC OSCILLATOR

A. Phase Shift Symmetry

Before analyzing the main SU(N) symmetry, we look at a more straightforward symmetry to illustrate the concepts reviewed in the previous section. Note that the transformation

$$a_k \to e^{i\phi}a_k \tag{6}$$

for some global phase shift ϕ leaves the Hamiltonian (1) unchanged.

This symmetry corresponds to time translation symmetry, which is given by the unitary time evolution operator $\mathcal{U} = e^{i\mathcal{H}\phi}$. This is clearly a symmetry of Hamiltonian as $\mathcal{H}$ commutes with $\mathcal{U}$, and it acts on a_k through

$$a_k \to e^{-i\mathcal{H}\phi}a_k e^{i\mathcal{H}\phi}. \tag{7}$$

One can verify through direct computation that (6) and (7) are identical. As a further cross check, note that in the case of infinitesimal ϕ, a_k transforms according to (5) as

$$a_k \to a_k + i\phi[-\mathcal{H}, a_k] = a_k + i\phi a_k,$$

which is indeed the infinitesimal form of the phase shift transformation.

To elucidate the connection of the phase shift symmetry with time translation symmetry, note that in the Heisenberg picture (see [1]), we have

$$\begin{aligned} x_k(t) &= x_k \cos t + p_k \sin t \\ p_k(t) &= p_k \cos t - x_k \sin t, \end{aligned}$$

which leads to

$$a_k(t) = e^{-it}a_k,$$

which is exactly the phase shift transformation with $\phi = -t$.

B. SU(N) Symmetry

Consider a general transformation of the form

$$a_j \to M_{jk}a_k \tag{8}$$

where $\mathbf{M} \in \mathbb{C}^{N\times N}$ is some fixed matrix. It is straightforward to check that $\mathcal{H}$ is unchanged under (8) if and only if $\mathbf{M}$ is a unitary matrix, and we restrict further discussion to this unitary case. It is also straightforward to check that the commutator relations (2), (3) are also unchanged under this transformation.

These transformations include the global phase shift symmetry discussed above. The phase shift symmetry corresponds to the conserved quantity $\mathcal{H}$, which isn't interesting as it isn't a new conserved quantity. Thus, we restrict our attention to transformations where $\mathbf{M} \in \mathrm{SU}(N)$, as a general unitary matrix can be formed by composing an SU(N) matrix with a phase shift.

Note that the transformation (8) corresponds to some unitary operator $\mathcal{U}_{\mathbf{M}}$ through (4). By combining (8) and (4), it is straightforward to verify that $\mathcal{U}_{\mathbf{M}}\mathcal{U}_{\mathbf{N}} = \mathcal{U}_{\mathbf{MN}}$, so the Lie group of such transformations

$$\mathcal{G} := \{\mathcal{U}_{\mathbf{M}} : \mathbf{M} \in \mathrm{SU}(N)\}$$

is isomorphic to SU(N). We now shift our focus to the Lie Algebra $\mathfrak{g}$ corresponding to $\mathcal{G}$.

V. INDUCED REPRESENTATION OF THE LIE ALGEBRA OF SU(N)

By considering infinitesimal versions of (8), we can concretely write down the elements of $\mathfrak{g}$ as Hermitian operators acting on the Hilbert space span $\{|n_1, \ldots, n_N\rangle\}$. This is an infinite dimensional representation of the Lie algebra of SU(N). Our goal in this section is to write down a basis for $\mathfrak{g}$ in this representation. In order to do this, we need to first review the structure of the Lie algebra in the fundamental representation, i.e. the set of traceless Hermitian $N \times N$ matrices.

A. Generators of the Lie Algebra of SU(N) in the Fundamental Representation

Note that $e^{-iA} \in \mathrm{SU}(N)$ if and only if A is a Hermitian traceless $N \times N$ matrix, so the Lie algebra of SU(N) represented by $N \times N$ matrices (i.e. the fundamental representation) is simply the vector space of traceless Hermitian $N \times N$ matrices. Denote this Lie algebra by $\mathfrak{su}(N)$.

Let A_{ij} be the $N \times N$ matrix given by

$$(A_{ij})_{k\ell} = -\frac{1}{2}\delta_{ki}\delta_{\ell j} - \frac{1}{2}\delta_{kj}\delta_{\ell i} + \frac{1}{N}\delta_{ij}\delta_{k\ell} \tag{9}$$

for $1 \leq i \leq j \leq N$, and let B_{ij} be the $N \times N$ matrix given by

$$(B_{ij})_{k\ell} = \frac{i}{2}\delta_{ki}\delta_{\ell j} - \frac{i}{2}\delta_{kj}\delta_{\ell i} \tag{10}$$

for $1 \leq i < j \leq N$. It is straightforward to verify that the matrices A_{ij}, B_{ij} all together span $\mathfrak{su}(N)$. Furthermore, they are all linearly independent except for the relation $\sum_{i=1}^{N} A_{ii} = 0$. We take these matrices to be our "basis" for $\mathfrak{su}(N)$, where it is implicitly understood that there is one linear relation among the basis elements. As a quick check, the space spanned by these basis elements is

$$\frac{N(N+1)}{2} + \frac{N(N-1)}{2} - 1 = N^2 - 1,$$

which is indeed the dimension of SU(N).

We now derive the bracket relations by computing the commutators of these basis matrices. A straightforward computation reveals the nonzero commutators to be

$$[A_{ij}, A_{ik}] = -\frac{i}{2}B_{jk} \tag{11}$$

$$[A_{ij}, A_{ii}] = iB_{ij} \tag{12}$$

$$[B_{ij}, B_{ik}] = -\frac{i}{2}B_{jk} \tag{13}$$

$$[A_{ij}, B_{ik}] = \frac{i}{2}A_{jk} \tag{14}$$

$$[A_{ij}, B_{ij}] = \frac{i}{2}(A_{jj} - A_{ii}) \tag{15}$$

$$[A_{ii}, B_{ij}] = \frac{i}{2}A_{ij}, \tag{16}$$

where distinct indices are assumed to take distinct values, and where we have extended the definition of A_{ij} and B_{ij} from (9) and (10) to all pairs (i, j). Bracket relations that can be derived by swapping two indices are also omitted. These constitute the bracket relations of the Lie algebra $\mathfrak{su}(N)$.

B. Generators of the Lie algebra in the Induced Representation

Let S_{ij} be the operator given by

$$S_{ij} = -\frac{1}{2}\left(a_i^\dagger a_j + a_j^\dagger a_i\right) + \frac{1}{N}\delta_{ij}\sum_{k=1}^{N} a_k^\dagger a_k \tag{17}$$

for $1 \leq i \leq j \leq N$, and let Q_{ij} be the operator given by

$$Q_{ij} = \frac{1}{2i}\left(-a_i^\dagger a_j + a_j^\dagger a_i\right). \tag{18}$$

for $1 \leq i < j \leq N$. A straightforward computation reveals that

$$[S_{ij}, a_k] = -(A_{ij})_{kr}a_r$$

and

$$[Q_{ij}, a_k] = -(B_{ij})_{kr}a_r.$$

Note that the infinitesimal transformation $\mathcal{U} = e^{-iS_{ij}\theta}$ sends

$$a_k \to a_k + i\theta[S_{ij}, a_k] = a_k - i\theta(A_{ij})_{kr}a_r = \left(e^{-i\theta A_{ij}}\right)_{kr} a_r,$$

and similarly the infinitesimal transformation $\mathcal{U} = e^{-iQ_{ij}\theta}$ sends

$$a_k \to \left(e^{-i\theta B_{ij}}\right)_{kr} a_r.$$

Therefore, we see that the S_{ij} and Q_{ij} are generators for the Lie algebra $\mathfrak{g}$ since their infinitesimal transformations act on the vector $\mathbf{a}$ in the same way as the infinitesimal generators for $\mathfrak{su}(N)$. We can directly compute the commutators of these operators, and the nonzero ones are as follows, following the same convention as (11)-(16):

$$[S_{ij}, S_{ik}] = -\frac{i}{2}Q_{jk} \tag{19}$$

$$[S_{ij}, S_{ii}] = iQ_{ij} \tag{20}$$

$$[Q_{ij}, Q_{ik}] = -\frac{i}{2}Q_{jk} \tag{21}$$

$$[S_{ij}, Q_{ik}] = \frac{i}{2}S_{jk} \tag{22}$$

$$[S_{ij}, Q_{ij}] = \frac{i}{2}(S_{jj} - S_{ii}) \tag{23}$$

$$[S_{ii}, Q_{ij}] = \frac{i}{2}S_{ij}. \tag{24}$$

Note that these match the bracket relations of $\mathfrak{su}(N)$ that we computed in the previous section, so we have constructed a genuine representation of the Lie algebra using Hermitian operators that act on the Hilbert space of harmonic oscillator states.

C. Connection to Orbital Angular Momentum

Note that the span of the Q_{ij} is itself a Lie algebra, since by (21), we have $[Q, Q] \sim Q$. This Lie algebra is isomorphic to the Lie algebra of $\mathrm{SO}(N)$, and arises due to the fact that the harmonic oscillator Hamiltonian has rotational symmetry.

We can expand (18) in terms of position and momentum as

$$Q_{ij} = -i(x_i p_j - x_j p_i),$$

which shows that the Q_{ij} correspond to orbital angular momentum. One can use this to show that the infinitesimal transformation corresponding to Q_{ij} is a rotation of the ij-plane.

D. Subrepresentations of Fixed Energy

Since the transformations given by (8) preserve $\mathcal{H}$, the corresponding Lie algebra $\mathfrak{g}$ should commute with $\mathcal{H}$. Indeed, one can manually verify from (17) and (18) that $[S_{ij}, \mathcal{H}] = [Q_{ij}, \mathcal{H}] = 0$. Therefore, the Lie algebra acting on energy eigenstates preserves the energy, so there is a natural subrepresentation of $\mathfrak{su}(N)$ given by the action of $\mathfrak{g}$ on the Hilbert space

$$\mathfrak{H}_n := \{|n_1, \dots, n_N\rangle : n_1 + \cdots + n_N = n\}.$$

Remarkably, this representation is actually irreducible, which means that there is no smaller subspace of $\mathfrak{H}_n$ which $\mathfrak{g}$ fixes (see [3]). It turns out that this representation is the fully symmetric representation with highest weight $(n, 0, \dots, 0)$ (see [4]). As a cross check, it's dimension can be computed by Weyl's character formula to be $\binom{n+N-1}{N-1}$, which exactly matches the degeneracy of the energy level.

VI. CONCLUSION

In this paper, we showed that the N-dimensional quantum harmonic oscillator possesses as $\mathrm{SU}(N)$ symmetry, which leads to conservation of the $\binom{N}{2}$ components of the angular momentum, as well as another $\frac{N(N+1)}{2} - 1$ conserved angular momentum like quantities. We further used the harmonic oscillator Hilbert space to generate a set of symmetric representations of $\mathrm{SU}(N)$.

These invariants have potential to shed light on systems in quantum field theory (QFT), since free fields in QFT are described by infinite dimensional oscillators. Furthermore, we were able to extract concrete representations of useful representations of $\mathrm{SU}(N)$ using the Hilbert space spanned by the energy eigenstates of the harmonic oscillator, which furthers our understanding of the representation theory of $\mathrm{SU}(N)$. In general, there is potential for physics to help inform representation theory, if we can find natural physical systems whose symmetry groups are given by the desired group.

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