ESTIMATION

GOPAL GOEL

Themes

- Find the right quantitative proxy.
- Average local information into global conclusions.
- Build a certificate or extremizer.
- Compare to the right asymptotic model.

1. Problems

(1) **[Putnam 2012 B4]**
Let $a_0 = 1$ and

$$a_{n+1} = a_n + e^{-a_n}$$

for $n \geq 0$. Does $a_n - \log n$ have a finite limit as $n \to \infty$?
Solution sketch

(2) **[TSTST 2024/1]**
For every pair of integers (i, j), choose a point $P_{i,j}$ inside the unit square

$$i < x < i+1, \qquad j < y < j+1.$$

Determine all real numbers $c > 0$ for which it is possible to choose the points so that every quadrilateral

$$P_{i,j}P_{i+1,j}P_{i+1,j+1}P_{i,j+1}$$

has perimeter strictly less than c.
Solution sketch

(3) **[IMO Shortlist 2018 A4]**
Let $a_0 = 0$ and $a_1 = 1$. For every $n \geq 2$, the number a_n is defined to be the average of some nonempty suffix of the preceding terms; that is, for some $1 \leq k \leq n$,

$$a_n = \frac{a_{n-1} + a_{n-2} + \cdots + a_{n-k}}{k}.$$

Find the largest possible value of $a_{2018} - a_{2017}$.
Solution sketch

(4) **[Putnam 2016 B2]**
A positive integer n is called squarish if either n is a perfect square or the distance from n to the nearest perfect square is a perfect square. Let $S(N)$ be the number of squarish integers between 1 and N. Find positive constants α, β such that

$$\frac{S(N)}{N^\alpha} \to \beta,$$

or prove that no such constants exist.

Solution sketch

(5) **[TST 2020/1]**

Choose positive integers $b_1, b_2, \ldots$ satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \cdots.$$

Let r be the largest real number such that

$$\frac{b_n}{n^2} \geq r$$

for all positive integers n. Determine all possible values of r.

Solution sketch

(6) **[Putnam 2014 A4]**

Let X be a random variable taking values in the nonnegative integers. Suppose

$$\mathbb{E}[X] = 1, \qquad \mathbb{E}[X^2] = 2, \qquad \mathbb{E}[X^3] = 5.$$

Find the minimum possible value of $\Pr(X = 0)$.

Solution sketch

(7) **[IMO Shortlist 2024 A3]**

Let $a_1, a_2, \ldots$ be an infinite sequence of positive real numbers. Prove that there exists a positive integer n such that

$$\frac{3^{a_1} + \cdots + 3^{a_n}}{(2^{a_1} + \cdots + 2^{a_n})^2} < \frac{1}{2024}.$$

Solution sketch

(8) **[Putnam 2020 A3]**

Let

$$a_0 = \frac{\pi}{2}, \qquad a_n = \sin(a_{n-1})$$

for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

Solution sketch

(9) **[TSTST 2015/1]**

Let $a_1, \ldots, a_n$ be real numbers written cyclically, and let $m < n$ be a positive integer. Call an index k good if for some $1 \leq \ell \leq m$,

$$a_k + a_{k+1} + \cdots + a_{k+\ell-1} \geq 0,$$

where indices are taken modulo n. Let T be the set of good indices. Prove that

$$\sum_{k \in T} a_k \geq 0.$$

Solution sketch

(10) **[Putnam 2012 B2]**

Let P be a fixed polyhedron. Suppose n balls have total volume V and cover the entire surface of P. Prove that

$$n > \frac{c(P)}{V^2}$$

for some positive constant $c(P)$ depending only on P.

Solution sketch

(11) **[IMO Shortlist 2017 A4]**

A sequence of real numbers $a_1, a_2, \ldots$ satisfies

$$a_n = -\max_{i+j=n}(a_i + a_j)$$

for all $n > 2017$. Prove that the sequence is bounded; that is, prove that there exists a constant M such that

$$|a_n| \leq M$$

for all positive integers n.

Solution sketch

(12) **[Putnam 2014 B2]**

Let f be a real-valued function on $[1,3]$ such that

$$-1 \leq f(x) \leq 1$$

for all x and

$$\int_1^3 f(x)\,dx = 0.$$

Find the maximum possible value of

$$\int_1^3 \frac{f(x)}{x}\,dx.$$

Solution sketch

2. Solution Sketches

1. [Putnam 2012 B4]. *Back to problem*
Problem.
Let $a_0 = 1$ and

$$a_{n+1} = a_n + e^{-a_n}$$

for $n \geq 0$. Does $a_n - \log n$ have a finite limit as $n \to \infty$?
Solution sketch.
Yes.
First prove by induction that

$$a_n > \log(n+1)$$

for every $n \geq 0$. This is true for $n = 0$. If $a_n > \log(n+1)$, then

$$a_{n+1} = a_n + e^{-a_n} > \log(n+1) + \frac{1}{n+1}.$$

Since

$$\log(n+2) - \log(n+1) = \log\left(1 + \frac{1}{n+1}\right) < \frac{1}{n+1},$$

we get

$$a_{n+1} > \log(n+2).$$

Now define

$$d_n = a_n - \log n$$

for $n \geq 1$. The inequality above shows $d_n > 0$. Also

$$d_{n+1} - d_n = e^{-a_n} + \log\frac{n}{n+1}.$$

Since $a_n > \log(n+1)$,

$$e^{-a_n} < \frac{1}{n+1}.$$

Therefore

$$d_{n+1} - d_n < \frac{1}{n+1} + \log\frac{n}{n+1} \leq 0.$$

Thus (d_n) is decreasing and bounded below by 0, so it converges to a finite limit.

2. [TSTST 2024/1]. *Back to problem*
Problem.
For every pair of integers (i, j), choose a point $P_{i,j}$ inside the unit square

$$i < x < i+1, \qquad j < y < j+1.$$

Determine all real numbers $c > 0$ for which it is possible to choose the points so that every quadrilateral

$$P_{i,j}P_{i+1,j}P_{i+1,j+1}P_{i,j+1}$$

has perimeter strictly less than c.

Solution sketch.

The answer is

$$c \geq 4.$$

First show that $c < 4$ is impossible. Look at an $n \times n$ block of selected points and average the perimeters of all elementary quadrilaterals in that block. For each fixed row, the sum of horizontal distances between consecutive selected points is at least the distance from the first selected point to the last selected point in that row. These points lie in squares whose x-coordinates are separated by about n, so the row contribution is greater than $n-2$.

Doing the analogous estimate in all rows and columns, and then averaging over the $(n-1)^2$ quadrilaterals, gives some quadrilateral with perimeter at least

$$4 - \frac{4}{n-1}.$$

Letting n be large rules out every $c < 4$.

For $c = 4$, choose points in separated-coordinate form

$$P_{i,j} = (f(i), f(j)),$$

where

$$i < f(i) < i + 1$$

and

$$|f(i+1) - f(i)| < 1$$

for every integer i. Such an f is easy to build by putting $f(i)$ close to $i+1/2$ and making adjacent increments slightly less than 1. Then each elementary quadrilateral has side lengths

$$|f(i+1) - f(i)|, \qquad |f(j+1) - f(j)|,$$

counted twice, so its perimeter is strictly less than 4. Any $c > 4$ is also possible.

3. [IMO Shortlist 2018 A4]. *Back to problem*

Problem.

Let $a_0 = 0$ and $a_1 = 1$. For every $n \geq 2$, the number a_n is defined to be the average of some nonempty suffix of the preceding terms; that is, for some $1 \leq k \leq n$,

$$a_n = \frac{a_{n-1} + a_{n-2} + \cdots + a_{n-k}}{k}.$$

Find the largest possible value of $a_{2018} - a_{2017}$.

Solution sketch.

The answer is

$$\frac{2016}{2017^2}.$$

For each n and $1 \leq k \leq n$, define

$$A(n,k) = \frac{a_{n-1} + \cdots + a_{n-k}}{k},$$

and set

$$M_n = \max_k A(n,k), \qquad m_n = \min_k A(n,k), \qquad \Delta_n = M_n - m_n.$$

The next term a_n is one of the numbers $A(n,k)$, and a_{n-1} also lies between m_n and M_n. Hence

$$a_n - a_{n-1} \le \Delta_n.$$

The key point is that the interval of possible suffix averages contracts. One proves directly from the definitions that

$$\Delta_{n+1} \le \frac{n}{n+1}\Delta_n.$$

Indeed, every suffix average at time $n+1$ is an average involving a_n and an old suffix average. The possible values cannot spread by more than the old spread, and the extremal comparison gives the factor $n/(n+1)$.

Let q be the first index for which $a_q < 1$. Before q, all terms are equal to 1. Then the first drop must occur by averaging all previous terms, including $a_0 = 0$, so

$$a_q = \frac{q-1}{q}.$$

At that moment the spread of possible averages is

$$\Delta_{q+1} = \frac{q-1}{q^2}.$$

Telescoping the contraction from $q+1$ to 2018 gives

$$a_{2018} - a_{2017} \le \Delta_{2018} \le \frac{q-1}{q \cdot 2017} \le \frac{2016}{2017^2}.$$

The last expression is maximized at $q = 2017$. Equality is achieved by taking

$$a_1 = \cdots = a_{2016} = 1,$$

then

$$a_{2017} = \frac{a_{2016} + \cdots + a_0}{2017} = \frac{2016}{2017},$$

and

$$a_{2018} = \frac{a_{2017} + a_{2016} + \cdots + a_1}{2017}.$$

This gives

$$a_{2018} - a_{2017} = \frac{2016}{2017^2}.$$

4. [Putnam 2016 B2]. *Back to problem*
Problem.

A positive integer n is called squarish if either n is a perfect square or the distance from n to the nearest perfect square is a perfect square. Let $S(N)$ be the number of squarish integers between 1 and N. Find positive constants α, β such that

$$\frac{S(N)}{N^\alpha} \to \beta,$$

or prove that no such constants exist.
Solution sketch.

The answer is

$$\alpha = \frac{3}{4}, \qquad \beta = \frac{4}{3}.$$

Consider integers nearest to k^2. The integers closer to k^2 than to any other square lie approximately in

$$[k^2 - k, k^2 + k].$$

Within this interval, the squarish integers are k^2 itself, plus those at square distances

$$1^2, 2^2, 3^2, \ldots$$

from k^2 on either side, as long as they remain in the interval. Thus the number contributed near k^2 is

$$1 + \lfloor \sqrt{k-1} \rfloor + \lfloor \sqrt{k} \rfloor,$$

up to harmless endpoint conventions.
For $k \le \sqrt{N}$, this is asymptotic to $2\sqrt{k}$. Therefore

$$S(N) \sim \sum_{k \le \sqrt{N}} 2\sqrt{k}.$$

Using Riemann sums,

$$\sum_{k \le \sqrt{N}} 2\sqrt{k} \sim \int_0^{\sqrt{N}} 2\sqrt{x}\, dx = \frac{4}{3} N^{3/4}.$$

The floor and endpoint errors contribute only $O(\sqrt{N})$, which is smaller than $N^{3/4}$. Hence

$$\frac{S(N)}{N^{3/4}} \to \frac{4}{3}.$$

5. [TST 2020/1]. *Back to problem*
Problem.

Choose positive integers $b_1, b_2, \ldots$ satisfying

$$1 = \frac{b_1}{1^2} > \frac{b_2}{2^2} > \frac{b_3}{3^2} > \cdots.$$

Let r be the largest real number such that

$$\frac{b_n}{n^2} \ge r$$

for all positive integers n. Determine all possible values of r.

Solution sketch.

The answer is

$$0 \leq r \leq \frac{1}{2}.$$

First prove the upper bound. We claim by induction that

$$b_n \leq \frac{n(n+1)}{2}.$$

This is true for $n = 1$. If it holds for n, then

$$\frac{b_{n+1}}{(n+1)^2} < \frac{b_n}{n^2} \leq \frac{n+1}{2n}.$$

Hence

$$b_{n+1} < \frac{(n+1)^3}{2n}.$$

But

$$\frac{(n+1)^3}{2n} = \frac{(n+1)(n+2)}{2} + \frac{n+1}{2n},$$

and $0 < \frac{n+1}{2n} < 1$ for $n \geq 2$. Since b_{n+1} is an integer, this gives

$$b_{n+1} \leq \frac{(n+1)(n+2)}{2}.$$

Thus

$$r \leq \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}.$$

The value $r = 1/2$ is attained by

$$b_n = \frac{n(n+1)}{2},$$

because

$$\frac{b_n}{n^2} = \frac{n+1}{2n}$$

is strictly decreasing and tends to $1/2$. The value $r = 0$ is easy, for example by taking any sequence whose ratios decrease to 0.

For $0 < r < 1/2$, take N large and for $n \geq N$ define

$$b_n = \lceil rn^2 \rceil + n.$$

Then $b_n/n^2 \to r$ from above. Also, for large n,

$$\frac{b_n}{n^2} \geq r + \frac{1}{n},$$

while

$$\frac{b_{n+1}}{(n+1)^2} \leq r + \frac{1}{n+1} + \frac{1}{(n+1)^2} < r + \frac{1}{n}.$$

So the ratios are strictly decreasing eventually. Choose the finitely many earlier b_n to splice into this tail, keeping the ratios strictly decreasing and positive. This realizes every $r \in [0, 1/2]$.

6. [Putnam 2014 A4]. *Back to problem*

Problem.

Let X be a random variable taking values in the nonnegative integers. Suppose

$$\mathbb{E}[X] = 1, \qquad \mathbb{E}[X^2] = 2, \qquad \mathbb{E}[X^3] = 5.$$

Find the minimum possible value of $\Pr(X = 0)$.

Solution sketch.

The answer is

$$\frac{1}{3}.$$

The key is to find a polynomial certificate. For every integer $m \geq 1$,

$$m^3 - 6m^2 + 11m = 6 + (m-1)(m-2)(m-3) \geq 6,$$

with the small values checked directly. Also this polynomial is 0 at $m = 0$. Therefore

$$\mathbb{E}[X^3 - 6X^2 + 11X] \geq 6 \Pr(X \geq 1).$$

The left side equals

$$5 - 12 + 11 = 4.$$

So

$$\Pr(X \geq 1) \leq \frac{2}{3},$$

and hence

$$\Pr(X = 0) \geq \frac{1}{3}.$$

Equality occurs for

$$\Pr(X = 0) = \frac{1}{3}, \qquad \Pr(X = 1) = \frac{1}{2}, \qquad \Pr(X = 3) = \frac{1}{6}.$$

These probabilities give the required first three moments.

7. [IMO Shortlist 2024 A3]. *Back to problem*

Problem.

Let $a_1, a_2, \ldots$ be an infinite sequence of positive real numbers. Prove that there exists a positive integer n such that

$$\frac{3^{a_1} + \cdots + 3^{a_n}}{(2^{a_1} + \cdots + 2^{a_n})^2} < \frac{1}{2024}.$$

Solution sketch.

Let

$$R_n = \frac{\sum_{i=1}^n 3^{a_i}}{\left(\sum_{i=1}^n 2^{a_i}\right)^2}, \qquad M_n = \max(a_1, \ldots, a_n).$$

We first prove

$$R_n \le \left(\frac{3}{4}\right)^{M_n}.$$

Indeed, since $a_i \le M_n$,

$$3^{a_i} = 2^{a_i}\left(\frac{3}{2}\right)^{a_i} \le 2^{a_i}\left(\frac{3}{2}\right)^{M_n}.$$

Summing gives

$$\sum_{i=1}^{n} 3^{a_i} \le \left(\frac{3}{2}\right)^{M_n} \sum_{i=1}^{n} 2^{a_i}.$$

Also

$$\sum_{i=1}^{n} 2^{a_i} \ge 2^{M_n}.$$

Hence

$$R_n \le \frac{(3/2)^{M_n}}{2^{M_n}} = \left(\frac{3}{4}\right)^{M_n}.$$

Choose μ such that

$$\left(\frac{3}{4}\right)^{\mu} < \frac{1}{2024}.$$

If some $M_n > \mu$, then $R_n < 1/2024$, and we are done. Otherwise, all $a_i \le \mu$. Then for every n,

$$\sum_{i=1}^{n} 3^{a_i} \le n3^{\mu}, \qquad \sum_{i=1}^{n} 2^{a_i} \ge n,$$

so

$$R_n \le \frac{3^{\mu}}{n}.$$

For large n, this is less than $1/2024$. Thus some n always works.

8. [Putnam 2020 A3]. *Back to problem*

Problem.

Let

$$a_0 = \frac{\pi}{2}, \qquad a_n = \sin(a_{n-1})$$

for $n \ge 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

Solution sketch.

It diverges.

The sequence is positive and decreasing after $a_1 = 1$. For $0 \le x \le 1$,

$$\sin x \geq x - \frac{x^3}{6}.$$

We prove by induction that

$$a_n \geq \frac{1}{\sqrt{n}}$$

for $n \geq 1$. The base case $a_1 = 1$ is equality. If $a_n \geq 1/\sqrt{n}$, then since $\sin x$ is increasing on $[0, 1]$,

$$a_{n+1} = \sin(a_n) \geq \sin \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} - \frac{1}{6n^{3/2}}.$$

A direct algebra check shows

$$\frac{1}{\sqrt{n}} - \frac{1}{6n^{3/2}} \geq \frac{1}{\sqrt{n+1}}.$$

Thus

$$a_n^2 \geq \frac{1}{n}$$

for all $n \geq 1$. Since the harmonic series diverges, so does

$$\sum_{n=1}^{\infty} a_n^2.$$

9. [TSTST 2015/1]. *Back to problem*

Problem.

Let $a_1, \ldots, a_n$ be real numbers written cyclically, and let $m < n$ be a positive integer. Call an index k good if for some $1 \leq \ell \leq m$,

$$a_k + a_{k+1} + \cdots + a_{k+\ell-1} \geq 0,$$

where indices are taken modulo n. Let T be the set of good indices. Prove that

$$\sum_{k \in T} a_k \geq 0.$$

Solution sketch.

For each index k, define

$$M(k) = \max_{1 \leq \ell \leq m} (a_k + a_{k+1} + \cdots + a_{k+\ell-1}),$$

and

$$N(k) = \max(M(k), 0).$$

Thus k is good exactly when $N(k) > 0$ or $M(k) = 0$; morally, $N(k)$ records the largest nonnegative forward block sum starting at k.

If k is good and ℓ attains $M(k)$, then

$$M(k) = a_k + (a_{k+1} + \cdots + a_{k+\ell-1}) \leq a_k + M(k+1) \leq a_k + N(k+1).$$

The zero case is harmless, so we obtain

$$a_k \geq N(k) - N(k+1)$$

for every good k.

Summing over good indices gives

$$\sum_{k\in T} a_k \geq \sum_{k\in T} N(k) - \sum_{k\in T} N(k+1).$$

The shifted sum cannot exceed the unshifted one: if $N(k+1) > 0$, then $k+1$ is good and the positive term $N(k+1)$ appears in the first sum; if $N(k+1) = 0$, it contributes nothing. Hence the right-hand side is nonnegative, proving the claim.

10. [Putnam 2012 B2]. *Back to problem*

Problem.

Let P be a fixed polyhedron. Suppose n balls have total volume V and cover the entire surface of P. Prove that

$$n > \frac{c(P)}{V^2}$$

for some positive constant $c(P)$ depending only on P.

Solution sketch.

Let the radii of the balls be $r_1, \dots, r_n$. Their total volume is a constant multiple of

$$\sum r_i^3.$$

Since P is fixed and polyhedral, there is a constant $C(P)$ such that the area of the part of the surface of P covered by a ball of radius r is at most

$$C(P)r^2.$$

This is clear locally on each face, and large balls can be absorbed into the constant.

Let $A(P) > 0$ be the surface area of P. Since the balls cover the surface,

$$A(P) \leq C(P)\sum r_i^2.$$

Now use Hölder or the power mean inequality:

$$\sum r_i^2 \leq n^{1/3}\left(\sum r_i^3\right)^{2/3}.$$

But $\sum r_i^3$ is a constant multiple of V. Therefore

$$A(P) \leq C'(P)n^{1/3}V^{2/3}.$$

Rearranging gives

$$n \geq \frac{c(P)}{V^2}.$$

The exact constants are irrelevant; the exponent comes from area scaling like r^2 while volume scales like r^3.

11. [IMO Shortlist 2017 A4]. *Back to problem*

Problem.

A sequence of real numbers $a_1, a_2, \ldots$ satisfies

$$a_n = -\max_{i+j=n}(a_i + a_j)$$

for all $n > 2017$. Prove that the sequence is bounded; that is, prove that there exists a constant M such that

$$|a_n| \leq M$$

for all positive integers n.

Solution sketch.

Set

$$D = 2017.$$

For each n, define the running upper and lower envelopes

$$M_n = \max_{k<n} a_k, \qquad m_n = -\min_{k<n} a_k = \max_{k<n}(-a_k).$$

Both M_n and m_n are nondecreasing. It is enough to prove both are bounded.

Fix $n > D$. We first bound a_n in terms of M_n and m_n. By the defining relation, there are indices p, q with $p + q = n$ such that

$$a_n = -(a_p + a_q).$$

Since $a_p, a_q \leq M_n$, this gives

$$a_n \geq -2M_n. \tag{1}$$

For the upper bound, choose $k < n$ such that $a_k = M_n$. Then

$$a_n = -\max_{\ell<n}(a_{n-\ell} + a_\ell) \leq -(a_{n-k} + a_k) = -a_{n-k} - M_n \leq m_n - M_n. \tag{2}$$

Combining (1) and (2),

$$-2M_n \leq a_n \leq m_n - M_n. \tag{3}$$

This gives recursive control of the envelopes. Since m_{n+1} is the maximum of the old m_n and possibly $-a_n$, while M_{n+1} is the maximum of the old M_n and possibly a_n, (3) implies

$$m_n \leq m_{n+1} \leq \max(m_n, 2M_n),$$

and

$$M_n \leq M_{n+1} \leq \max(M_n, m_n - M_n). \tag{4}$$

Now call an index $n > D$ **lucky** if

$$m_n \leq 2M_n.$$

There are two cases.

Case 1: Some lucky index exists. Suppose n is lucky. Then (4) gives

$$m_{n+1} \leq 2M_n,$$

and also

$$M_{n+1} \le \max(M_n, m_n - M_n) \le \max(M_n, M_n) = M_n.$$

Since $M_{n+1} \ge M_n$, we get

$$M_{n+1} = M_n.$$

Also

$$m_{n+1} \le 2M_n = 2M_{n+1},$$

so $n+1$ is lucky too. Repeating this argument, every later index remains lucky and $M_k = M_n$ for all $k > n$. Hence for all later k,

$$m_k \le 2M_k = 2M_n,$$

so both envelopes are bounded.

Case 2: No lucky index exists. Then for every $n > D$,

$$m_n > 2M_n.$$

In particular, $2M_n < m_n$, so (4) gives

$$m_{n+1} \le m_n.$$

But $m_n \le m_{n+1}$, hence

$$m_{n+1} = m_n$$

for every $n > D$. Thus all m_n are bounded by m_{D+1}. Since $m_n > 2M_n$, we also have $M_n < m_n/2$, so the M_n are bounded too.

In both cases, m_n and M_n are bounded. Therefore the original sequence (a_n) is bounded.

12. [Putnam 2014 B2]. *Back to problem*

Problem.

Let f be a real-valued function on $[1, 3]$ such that

$$-1 \le f(x) \le 1$$

for all x and

$$\int_1^3 f(x)\,dx = 0.$$

Find the maximum possible value of

$$\int_1^3 \frac{f(x)}{x}\,dx.$$

Solution sketch.

The answer is

$$\log \frac{4}{3}.$$

Since $1/x$ is decreasing, positive mass should be placed as far left as possible and negative mass as far right as possible. The candidate optimizer is

$$g(x) = \begin{cases} 1, & 1 \le x \le 2, \\ -1, & 2 < x \le 3. \end{cases}$$

It has mean zero, and its value is

$$\int_1^2 \frac{dx}{x} - \int_2^3 \frac{dx}{x} = \log 2 - \log \frac{3}{2} = \log \frac{4}{3}.$$

To prove optimality, compare any admissible f with g. Let

$$h = g - f.$$

Then

$$\int_1^3 h(x)\,dx = 0.$$

On $[1,2]$, $h \ge 0$, and on $[2,3]$, $h \le 0$. Since $1/x \ge 1/2$ on $[1,2]$ and $1/x \le 1/2$ on $[2,3]$,

$$\int_1^3 \frac{h(x)}{x}\,dx \ge \frac{1}{2}\int_1^3 h(x)\,dx = 0.$$

Therefore

$$\int_1^3 \frac{f(x)}{x}\,dx \le \int_1^3 \frac{g(x)}{x}\,dx = \log \frac{4}{3}.$$