ANALYSIS

GOPAL GOEL

1. Problems

(1) **[IMO Shortlist 2018 A4]**
Let $a_0 = 0$ and $a_1 = 1$. For every $n \geq 2$, the number a_n is defined to be the average of some nonempty suffix of the preceding terms; that is, for some $1 \leq k \leq n$,

$$a_n = \frac{a_{n-1} + a_{n-2} + \cdots + a_{n-k}}{k}.$$

Find the largest possible value of $a_{2018} - a_{2017}$.

Solution sketch

(2) **[IMO Shortlist 2017 A4]**
A sequence of real numbers $a_1, a_2, \ldots$ satisfies

$$a_n = -\max_{i+j=n}(a_i + a_j)$$

for all $n > 2017$. Prove that the sequence is bounded; that is, prove that there exists a constant M such that

$$|a_n| \leq M$$

for all positive integers n.

Solution sketch

(3) **[Putnam 2010 A6]**
Let $f : [0, \infty) \to \mathbb{R}$ be positive, continuous, decreasing, and satisfy

$$\lim_{x \to \infty} f(x) = 0.$$

Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\, dx$$

diverges.

Solution sketch

(4) **[TST 2021/3]**
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers $x < y < z$,

$$f(y) - \left(\frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z)\right) \leq f\left(\frac{x+z}{2}\right) - \frac{f(x) + f(z)}{2}.$$

Solution sketch

(5) **[Putnam 2010 B5]**
Does there exist a strictly increasing differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = f(f(x))$$

for all real numbers x?

Solution sketch

(6) **[TSTST 2018/9]**
Show that there is an absolute constant $c < 1$ with the following property. Whenever P is a polygon of area 1 in the plane, one can translate it by a distance $1/100$ in some direction to obtain a polygon Q for which the intersection of the interiors of P and Q has total area at most c.

Solution sketch

(7) **[Putnam 2011 B5]**
Let $a_1, \dots, a_n$ be real numbers. Suppose that for some $A > 0$,

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{1 + (x - a_i)^2} \right)^2 dx \leq An.$$

Prove that there exists $B > 0$, depending only on A, such that

$$\sum_{i,j=1}^{n} \left(1 + (a_i - a_j)^2\right) \geq Bn^3.$$

Solution sketch

(8) **[Putnam 2019 A6]**
Let g be continuous on $[0, 1]$ and twice differentiable on $(0, 1)$. Suppose that for some real $r > 1$,

$$\lim_{x \to 0^+} \frac{g(x)}{x^r} = 0.$$

Prove that either

$$\lim_{x \to 0^+} g'(x) = 0$$

or

$$\limsup_{x \to 0^+} x^r |g''(x)| = \infty.$$

Solution sketch

(9) **[All-Russian MO 2018 G11/7]**
Let

$$a_n = \left\lfloor n^{2018/2017} \right\rfloor$$

for $n = 1, 2, 3, \dots$. Prove that there exists a positive integer N such that among any N consecutive terms of the sequence, there is a term whose decimal representation contains the digit 5.

Solution sketch

2. Solution Sketches

1. [IMO Shortlist 2018 A4]. *Back to problem*
Problem.

Let $a_0 = 0$ and $a_1 = 1$. For every $n \geq 2$, the number a_n is defined to be the average of some nonempty suffix of the preceding terms; that is, for some $1 \leq k \leq n$,

$$a_n = \frac{a_{n-1} + a_{n-2} + \cdots + a_{n-k}}{k}.$$

Find the largest possible value of $a_{2018} - a_{2017}$.

Solution sketch.

The answer is

$$\frac{2016}{2017^2}.$$

For each n and $1 \leq k \leq n$, define

$$A(n,k) = \frac{a_{n-1} + \cdots + a_{n-k}}{k},$$

and set

$$M_n = \max_k A(n,k), \qquad m_n = \min_k A(n,k), \qquad \Delta_n = M_n - m_n.$$

The next term a_n is one of the numbers $A(n,k)$, and a_{n-1} also lies between m_n and M_n. Hence

$$a_n - a_{n-1} \leq \Delta_n.$$

The key point is that the interval of possible suffix averages contracts. One proves directly from the definitions that

$$\Delta_{n+1} \leq \frac{n}{n+1}\Delta_n.$$

Indeed, every suffix average at time $n+1$ is an average involving a_n and an old suffix average. The possible values cannot spread by more than the old spread, and the extremal comparison gives the factor $n/(n+1)$.

Let q be the first index for which $a_q < 1$. Before q, all terms are equal to 1. Then the first drop must occur by averaging all previous terms, including $a_0 = 0$, so

$$a_q = \frac{q-1}{q}.$$

At that moment the spread of possible averages is

$$\Delta_{q+1} = \frac{q-1}{q^2}.$$

Telescoping the contraction from $q+1$ to 2018 gives

$$a_{2018} - a_{2017} \leq \Delta_{2018} \leq \frac{q-1}{q \cdot 2017} \leq \frac{2016}{2017^2}.$$

The last expression is maximized at $q = 2017$. Equality is achieved by taking

$$a_1 = \cdots = a_{2016} = 1,$$

then

$$a_{2017} = \frac{a_{2016} + \cdots + a_0}{2017} = \frac{2016}{2017},$$

and

$$a_{2018} = \frac{a_{2017} + a_{2016} + \cdots + a_1}{2017}.$$

This gives

$$a_{2018} - a_{2017} = \frac{2016}{2017^2}.$$

2. [IMO Shortlist 2017 A4]. *Back to problem*

Problem.

A sequence of real numbers $a_1, a_2, \ldots$ satisfies

$$a_n = -\max_{i+j=n}(a_i + a_j)$$

for all $n > 2017$. Prove that the sequence is bounded; that is, prove that there exists a constant M such that

$$|a_n| \le M$$

for all positive integers n.

Solution sketch.

Set

$$D = 2017.$$

For each n, define the running upper and lower envelopes

$$M_n = \max_{k<n} a_k, \qquad m_n = -\min_{k<n} a_k = \max_{k<n}(-a_k).$$

Both M_n and m_n are nondecreasing. It is enough to prove both are bounded.

Fix $n > D$. We first bound a_n in terms of M_n and m_n. By the defining relation, there are indices p, q with $p + q = n$ such that

$$a_n = -(a_p + a_q).$$

Since $a_p, a_q \le M_n$, this gives

$$a_n \ge -2M_n. \tag{1}$$

For the upper bound, choose $k < n$ such that $a_k = M_n$. Then

$$a_n = -\max_{\ell<n}(a_{n-\ell} + a_\ell) \le -(a_{n-k} + a_k) = -a_{n-k} - M_n \le m_n - M_n. \tag{2}$$

Combining (1) and (2),

$$-2M_n \le a_n \le m_n - M_n. \tag{3}$$

This gives recursive control of the envelopes. Since m_{n+1} is the maximum of the old m_n and possibly $-a_n$, while M_{n+1} is the maximum of the old M_n and possibly a_n, (3) implies

$$m_n \le m_{n+1} \le \max(m_n, 2M_n),$$

and

$$M_n \le M_{n+1} \le \max(M_n, m_n - M_n). \tag{4}$$

Now call an index $n > D$ **lucky** if

$$m_n \le 2M_n.$$

There are two cases.

Case 1: Some lucky index exists. Suppose n is lucky. Then (4) gives

$$m_{n+1} \le 2M_n,$$

and also

$$M_{n+1} \le \max(M_n, m_n - M_n) \le \max(M_n, M_n) = M_n.$$

Since $M_{n+1} \ge M_n$, we get

$$M_{n+1} = M_n.$$

Also

$$m_{n+1} \le 2M_n = 2M_{n+1},$$

so $n+1$ is lucky too. Repeating this argument, every later index remains lucky and $M_k = M_n$ for all $k > n$. Hence for all later k,

$$m_k \le 2M_k = 2M_n,$$

so both envelopes are bounded.

Case 2: No lucky index exists. Then for every $n > D$,

$$m_n > 2M_n.$$

In particular, $2M_n < m_n$, so (4) gives

$$m_{n+1} \le m_n.$$

But $m_n \le m_{n+1}$, hence

$$m_{n+1} = m_n$$

for every $n > D$. Thus all m_n are bounded by m_{D+1}. Since $m_n > 2M_n$, we also have $M_n < m_n/2$, so the M_n are bounded too.

In both cases, m_n and M_n are bounded. Therefore the original sequence (a_n) is bounded.

3. [Putnam 2010 A6]. *Back to problem*

Problem.

Let $f : [0, \infty) \to \mathbb{R}$ be positive, continuous, decreasing, and satisfy

$$\lim_{x\to\infty} f(x) = 0.$$

Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\, dx$$

diverges.

Solution sketch.

For integers $a < b$, split the integral over $[a, b]$ into unit intervals:

$$\int_a^b \frac{f(x) - f(x+1)}{f(x)}\,dx = \int_0^1 \sum_{k=a}^{b-1} \frac{f(x+k) - f(x+k+1)}{f(x+k)}\,dx.$$

Since f is decreasing, the numerators are nonnegative. Also $f(x+k) \le f(x+a)$, so

$$\sum_{k=a}^{b-1} \frac{f(x+k) - f(x+k+1)}{f(x+k)} \ge \frac{f(x+a) - f(x+b)}{f(x+a)}.$$

Choose $b > a$ so large that

$$f(b) < \frac{1}{2} f(a+1).$$

Then for $x \in [0, 1]$, we have $f(x+a) \ge f(a+1)$ and $f(x+b) \le f(b)$, so

$$\frac{f(x+a) - f(x+b)}{f(x+a)} \ge \frac{1}{2}.$$

Thus the integral over $[a, b]$ is at least $1/2$.
Now choose $a_0 = 0$ and recursively choose a_{r+1} so large that

$$f(a_{r+1}) < \frac{1}{2} f(a_r + 1).$$

The intervals $[a_r, a_{r+1}]$ are disjoint and each contributes at least $1/2$, so the total integral diverges.

4. [TST 2021/3]. *Back to problem*
Problem.
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers $x < y < z$,

$$f(y) - \left(\frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z) \right) \le f\left(\frac{x+z}{2} \right) - \frac{f(x) + f(z)}{2}.$$

Solution sketch.
The answer is

$$f(t) = at^2 + bt + c \qquad \text{with } a \le 0.$$

Put

$$m = \frac{x+z}{2}, \qquad h = \frac{z-x}{2}.$$

Then $x = m - h$ and $z = m + h$. After rearranging, the given inequality says that for every $y \in (m-h, m+h)$,

$$f(y) \le f(m) + \frac{f(m+h) - f(m-h)}{2h}(y - m).$$

Thus every symmetric secant slope through m gives an affine upper support for f near m. This is the analytic translation of the problem.
From this local supporting-line property, one first gets concavity. Indeed, if the graph of f rose above a chord in a way incompatible with concavity, then at a point of maximal excess over that chord, no affine line through the point could remain above nearby values on both sides. The

support-line condition rules this out. Hence f is concave, so in particular f is continuous and has one-sided derivatives everywhere.

For a concave function, the possible supporting slopes at m form the interval

$$[f'_+(m), f'_-(m)].$$

But the slope

$$\frac{f(m+h)-f(m-h)}{2h}$$

is a supporting slope for every $h > 0$. At every point m where f is differentiable, this slope must therefore equal $f'(m)$, independent of h. Hence for almost every m and every $h > 0$,

$$f(m+h) - f(m-h) = 2hf'(m).$$

Comparing this identity on overlapping arithmetic progressions gives zero third finite difference:

$$f(x+3d) - 3f(x+2d) + 3f(x+d) - f(x) = 0.$$

By continuity, this holds for all x, d. A continuous function with vanishing third finite difference on every arithmetic progression is a quadratic polynomial. Concavity forces the quadratic coefficient to be nonpositive.

Conversely, every linear function and every downward-facing quadratic satisfies the condition: for such a function, the symmetric secant slope equals the tangent slope at the midpoint, and concavity places the graph below its tangent line.

5. [Putnam 2010 B5]. *Back to problem*
Problem.

Does there exist a strictly increasing differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = f(f(x))$$

for all real numbers x?

Solution sketch.

No such function exists.

Assume such an f exists. Since f is strictly increasing and differentiable, we have $f'(x) \geq 0$. But

$$f'(x) = f(f(x)).$$

If $y > x$, then $f(y) > f(x)$, and applying the increasing function f again gives

$$f(f(y)) > f(f(x)).$$

Thus f' is strictly increasing. Since $f' \geq 0$ everywhere and is strictly increasing, actually $f'(x) > 0$ for every x; otherwise values to the left would have negative derivative.

Now $f' = f \circ f$ is differentiable, so f is twice differentiable. Differentiating gives

$$f''(x) = f'(f(x))f'(x).$$

Both factors are positive and increasing, hence f'' is positive and increasing. Choose M and $\alpha > 0$ such that

$$f''(x) > 4\alpha$$

for all $x \geq M$. Integrating twice, for all sufficiently large x, say $x \geq M_0$, we get a quadratic lower bound

$$f(x) \geq f(M) + f'(M)(x - M) + 2\alpha(x - M)^2 \geq \alpha x^2.$$

Now take T large enough so that $T \geq M_0$ and $\alpha T^2 > M_0$. Then for every $x \geq T$, we have $f(x) > M_0$, so the same quadratic lower bound can be applied to $f(x)$. Hence

$$f'(x) = f(f(x)) \geq \alpha f(x)^2.$$

Therefore

$$\int_T^{2T} \frac{f'(x)}{f(x)^2}\, dx \geq \alpha T.$$

But the left-hand side is exactly

$$\left[-\frac{1}{f(x)}\right]_T^{2T} = \frac{1}{f(T)} - \frac{1}{f(2T)} \leq \frac{1}{f(T)}.$$

As $T \to \infty$, the right side αT tends to infinity, while $1/f(T) \to 0$, since $f(T) \geq \alpha T^2$. This contradiction proves that no such function exists.

6. [TSTST 2018/9]. *Back to problem*

Problem.

Show that there is an absolute constant $c < 1$ with the following property. Whenever P is a polygon of area 1 in the plane, one can translate it by a distance $1/100$ in some direction to obtain a polygon Q for which the intersection of the interiors of P and Q has total area at most c.

Solution sketch.

The polygon structure is not important; only measurability and area matter. Let S be the set of area 1 and put

$$d = \frac{1}{100}.$$

Suppose for contradiction that for every vector v with $|v| = d$,

$$|S \cap (S + v)| > 1 - \varepsilon.$$

Equivalently, each small translation loses area less than ε.

Now take any sequence of N jumps $v_1, \ldots, v_N$, each of length d, and put

$$t_k = v_1 + \cdots + v_k.$$

Passing from $S + t_{k-1}$ to $S + t_k$ loses at most ε area. By the union bound,

$$|S \cap (S + t_1) \cap \cdots \cap (S + t_N)| \geq 1 - N\varepsilon.$$

In particular,

$$|S \cap (S + t_N)| \geq 1 - N\varepsilon.$$

Take $N = 800$. Since

$$800 \cdot \frac{1}{100} = 8,$$

any vector w with $|w| \leq 8$ can be written as a sum of 800 vectors, each of length $1/100$. Hence the assumption implies

$$|S \cap (S + w)| \geq 1 - 800\varepsilon$$

for every $|w| \le 8$.

Now average over w chosen uniformly from the disk D of radius 8. On one hand, the previous inequality makes the average overlap at least $1 - 800\varepsilon$. On the other hand,

$$\mathbb{E}_w|S \cap (S+w)| = \int_{x\in S} \Pr(x \in S+w)\, dx.$$

For fixed x, the set of $w \in D$ such that $x \in S+w$ has area at most $|S| = 1$, while

$$|D| = 64\pi.$$

Thus

$$\Pr(x \in S+w) \le \frac{1}{64\pi},$$

and so

$$\mathbb{E}_w|S \cap (S+w)| \le \frac{1}{64\pi}.$$

Therefore

$$1 - 800\varepsilon \le \frac{1}{64\pi},$$

so

$$\varepsilon \ge \frac{1 - \frac{1}{64\pi}}{800} > 0.001.$$

Taking $c = 0.999$ works. The analytic idea is that a finite-area set cannot be almost invariant under all translations in a large disk.

7. [Putnam 2011 B5]. *Back to problem*

Problem.

Let $a_1, \ldots, a_n$ be real numbers. Suppose that for some $A > 0$,

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{1+(x-a_i)^2} \right)^2 dx \le An.$$

Prove that there exists $B > 0$, depending only on A, such that

$$\sum_{i,j=1}^{n} \left(1 + (a_i - a_j)^2\right) \ge Bn^3.$$

Solution sketch.

Let

$$K(x) = \frac{1}{1+x^2}.$$

Expanding the square gives

$$\sum_{i,j} \int_{-\infty}^{\infty} K(x-a_i)K(x-a_j)\, dx \le An.$$

The integral depends only on $a_i - a_j$. There is an absolute constant $c > 0$ such that for every real y,

$$\int_{-\infty}^{\infty} K(x)K(x-y)\,dx \geq \frac{c}{1+y^2}.$$

Hence

$$\sum_{i,j} \frac{1}{1+(a_i-a_j)^2} \leq CAn$$

for an absolute constant C.

Now apply Cauchy's inequality to the n^2 positive numbers

$$u_{ij} = 1 + (a_i - a_j)^2.$$

We have

$$\left(\sum u_{ij}\right)\left(\sum \frac{1}{u_{ij}}\right) \geq n^4.$$

Since the second factor is at most CAn, the first factor is at least

$$\frac{n^4}{CAn} = Bn^3.$$

This proves the desired estimate.

Build a Certificate

Problems where the proof is driven by a supporting object: a tight certificate, convex envelope, extremizer, or weighted counting inequality.

8. [Putnam 2019 A6]. *Back to problem*

Problem.

Let g be continuous on $[0,1]$ and twice differentiable on $(0,1)$. Suppose that for some real $r > 1$,

$$\lim_{x\to 0^+} \frac{g(x)}{x^r} = 0.$$

Prove that either

$$\lim_{x\to 0^+} g'(x) = 0$$

or

$$\limsup_{x\to 0^+} x^r |g''(x)| = \infty.$$

Solution sketch.

Assume the second alternative fails. Then there is some M such that, near 0,

$$|g''(x)| \leq Mx^{-r}.$$

We prove $g'(x) \to 0$.

Suppose not. Then there are $\varepsilon_0 > 0$ and $x_n \to 0$ with

$$|g'(x_n)| > \varepsilon_0.$$

Fix a small $\varepsilon > 0$. Since $g(x) = o(x^r)$, choose δ so that

$$|g(x)| < \varepsilon x^r$$

for $x < \delta$.
For n large, $x_n < \delta/2$ and

$$h_n = \frac{\varepsilon_0 x_n^r}{2M} < x_n.$$

On the interval $[x_n, x_n + h_n]$, the derivative cannot change by more than $\varepsilon_0/2$, because

$$\int_{x_n}^{x_n+h_n} Mt^{-r}\,dt \le Mh_n x_n^{-r} = \frac{\varepsilon_0}{2}.$$

Thus $|g'| \ge \varepsilon_0/2$ throughout that interval, so

$$|g(x_n + h_n) - g(x_n)| \ge \frac{\varepsilon_0}{2} h_n = \frac{\varepsilon_0^2}{4M} x_n^r.$$

But both endpoints are less than δ, and $g(x) = o(x^r)$, so

$$|g(x_n + h_n) - g(x_n)| \le \varepsilon((x_n + h_n)^r + x_n^r) \le \varepsilon(2^r + 1)x_n^r.$$

Since ε was arbitrary, this is impossible. Therefore $g'(x) \to 0$.

9. [All-Russian MO 2018 G11/7]. *Back to problem*
Problem.
Let

$$a_n = \left\lfloor n^{2018/2017} \right\rfloor$$

for $n = 1, 2, 3, \ldots$. Prove that there exists a positive integer N such that among any N consecutive terms of the sequence, there is a term whose decimal representation contains the digit 5.
Solution sketch.
The proof has two ingredients.
First, use a decimal-density lemma. For a sufficiently large fixed length L, every nonconstant arithmetic progression of L positive integers contains a number whose decimal representation has the digit 5. One way to formulate the reason is to work modulo a large power of 10: if the common difference is not divisible by that power, then the progression moves through enough residue classes to hit a residue with digit 5; if the common difference is divisible by that power, divide out powers of 10 and look at the higher decimal block. The exact value of L is irrelevant.
Second, show that every sufficiently long block of consecutive terms of

$$\lfloor n^\alpha \rfloor, \qquad \alpha = \frac{2018}{2017},$$

contains a nonconstant arithmetic progression of length L among its values. The reason is local linearization. For $0 \le t < L$, Taylor's formula gives

$$(n + tq)^\alpha = n^\alpha + tq\alpha n^{\alpha-1} + O(L^2 q^2 n^{\alpha-2}).$$

Since $\alpha - 2 < 0$, the quadratic error becomes small for large n on short windows. Choose q in a bounded range so that

$$q\alpha n^{\alpha-1}$$

is very close to an integer D. Then, for large n,

$$\lfloor (n + tq)^\alpha \rfloor = \lfloor n^\alpha \rfloor + tD$$

for $t = 0, 1, \ldots, L-1$. Thus the corresponding sequence values contain an arithmetic progression of length L.

By the decimal-density lemma, one of those values contains the digit 5. The finitely many small starting indices can be handled by enlarging N.