

CRITICAL GROUPS OF ITERATED CONES

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ABSTRACT. Let G be a finite graph, and let G_n be the n -th iterated cone over G . We study the structure of the critical group of G_n arising in divisor and sandpile theory.

1. INTRODUCTION

The critical group $\mathcal{K}(G)$ of a connected graph G is the torsion part of the cokernel of its discrete Laplacian (details appear below). It is known as the degree-zero part of the Picard group or as the Jacobian of G in the divisor theory of graphs ([1]). It is isomorphic to the sandpile group of G from statistical physics ([4]) and to the group of parking functions of G from combinatorics ([5]). The n -th iterated cone over G , denoted G_n , is the join of G and the complete graph on n -vertices, K_n , formed by connecting each vertex of G with each vertex of K_n by an undirected edge. Our main result is Theorem 1, which provides a description of the structure of $\mathcal{K}(G_n)$ as an abelian group.

The question of the structure of $\mathcal{K}(G_n)$ was addressed previously in [2]. In that paper, Theorem A provides a short exact sequence for $\mathcal{K}(G_n)$ (cf. our Corollary 4) and Corollary B computes the order of $\mathcal{K}(G)$ in terms of the characteristic polynomial of the Laplacian of G (cf. our Theorem 1 (3)). We give new short and direct proofs of both of these results. We also give a partial answer to Question 1.2 of [2], which asks when the short exact sequence splits. (See the discussion after Corollary 4, below.)

Acknowledgements. We are grateful to David Zureick-Brown for presenting the problem of determining the structure of $\mathcal{K}(G_n)$ to us. We thank Collin Perkinson for comments on the exposition. We also thank our anonymous referee for helpful suggestions.

2. MAIN RESULTS

Let G be an Eulerian digraph. As a special case, G could be an undirected graph. Loops and multiple edges are allowed. We assume that G is connected with finite vertex set V and finite edge multiset E . We write (v, w) for a directed edge starting at v and ending at w . The main Eulerian property we need is that the indegree and outdegree are equal at each vertex. Letting $\mathbb{Z}V$ denote the free abelian group on the vertices, the (discrete) Laplacian of G is the homomorphism $L: \mathbb{Z}V \rightarrow \mathbb{Z}V$ determined by $L(v) = \text{outdeg}(v)v - \sum_{(v,w) \in E} w$ for each $v \in V$. We assume the vertices are ordered so that we can identify L with a $k \times k$ matrix where $k := |V|$. Then $L = D - A^t$ where D is the diagonal matrix of the outdegrees of the vertices and A^t is the transpose of the directed adjacency matrix of G . The i, j -th entry of A is the number of edges from the i -th vertex to the j -th vertex. The image of L lies in the kernel of the “degree” homomorphism $\delta: \mathbb{Z}V \rightarrow \mathbb{Z}V$ determined by $\delta(v) = 1$

2010 *Mathematics Subject Classification.* primary 05C25, secondary 05C76.

Key words and phrases. graph Laplacian, critical group, Abelian sandpile, cone over a graph.

for each $v \in V$. The *critical group* of G is

$$\mathcal{K}(G) := \ker \delta / \operatorname{im} L.$$

Fixing any vertex $u \in V$, there is an isomorphism

$$\begin{aligned} \operatorname{cok}(L) &\rightarrow \mathcal{K}(G) \oplus \mathbb{Z} \\ f &\mapsto (f - \delta(f)\chi_u, \delta(f)), \end{aligned}$$

where $\operatorname{cok}(L)$ is the cokernel of L and $\chi_u \in \mathbb{Z}^V$ is the indicator function for u . It is well-known (e.g., via the matrix-tree theorem ([6, Thm. 5.6.8])) that since G is connected, the rank of L is $k - 1$, and hence $\mathcal{K}(G)$ is finite. Deleting the row and column corresponding to u from the matrix L gives the *reduced Laplacian* \tilde{L} of G , and since G is Eulerian ([3, Theorem 12.1]), there is an isomorphism

$$\mathcal{K}(G) \simeq \operatorname{cok}(\tilde{L})$$

over \mathbb{Z} .

Theorem 1. *Let G be an Eulerian digraph with k vertices and Laplacian L . Let G_n be the n -th cone over G where $n \geq 2$.*

- (1) *Let $\mathbf{1}$ be the $k \times k$ matrix whose entries are all 1, and let I_k be the $k \times k$ identity matrix. Then*

$$\mathcal{K}(G_n) \simeq (\mathbb{Z}/(n+k)\mathbb{Z})^{n-2} \oplus \operatorname{cok}(nI_k + L + \mathbf{1}).$$

- (2) *The group $\operatorname{cok}(nI_k + L + \mathbf{1})$ has a subgroup isomorphic to $\mathbb{Z}/(n+k)\mathbb{Z}$.*

- (3) ([2, Corollary B]) *The order of the critical group of G_n is*

$$|\mathcal{K}(G_n)| = \frac{|p_L(-n)|}{n} (n+k)^{n-1}$$

where p_L is the characteristic polynomial of L .

Proof. Order the vertices of G_n so that the cone vertices appear at the end. The reduced Laplacian for G_n is then, in block form,

$$\tilde{L}_n = \left[\begin{array}{c|c} nI_k + L & -\mathbf{1} \\ \hline -\mathbf{1} & (n+k)I_{n-1} - \mathbf{1} \end{array} \right],$$

where each $\mathbf{1}$ denotes a matrix of 1s (with dimensions inferred from context). Since G is Eulerian, all row and column sums of L are 0. Perform the following operations in order on \tilde{L}_n :

- (1) Subtract the last column from all other columns.
- (2) Add all but the last row to the last row.
- (3) Add the last row to all rows but the last.

The result is the block matrix

$$(1) \quad M := \left[\begin{array}{c|c|c} nI_k + L + \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & (n+k)I_{n-2} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & 1 \end{array} \right].$$

Then $\operatorname{cok}(M) \simeq \operatorname{cok}(\tilde{L}_n) \simeq \mathcal{K}(G_n)$, and Part 1 follows.

For Part 2, first note that since G_n is connected, \tilde{L}_n has full rank, and hence so does $A := nI_k + L + \mathbf{1}$. Consider the homomorphism $\phi: \mathbb{Z} \rightarrow \text{cok}(A)$ sending 1 to the all-ones vector $\vec{\mathbf{1}}$. If $\ell \in \ker \phi$, then there is a vector $\vec{v} \in \mathbb{Z}^k$ such that $A\vec{v} = \ell \cdot \vec{\mathbf{1}}$. However, $A\vec{\mathbf{1}} = (n+k) \cdot \vec{\mathbf{1}}$. Since A has full rank and \vec{v} is an integer vector, it follows that $n+k$ divides ℓ and \vec{v} is a constant vector. Hence, $\ker \phi$ is generated by $n+k$.

Finally, for Part 3, note that $|\mathcal{K}(G_n)| = \det(M) = (n+k)^{n-2} |\det(nI_k + L + \mathbf{1})|$. Let r_1, \dots, r_k be the rows of $nI_k + L$. For each $i = 1, \dots, k$, we use the identity $r_1 + \dots + r_k = n\vec{\mathbf{1}}$ to substitute for r_i and use the fact that the determinant is an alternating multilinear function of the rows of a matrix to get

$$p_L(-n) = \det(nI_k + L) = \det(r_1, \dots, r_k) = n \det(r_1, \dots, \underbrace{\vec{\mathbf{1}}}_i, \dots, r_k),$$

where $\vec{\mathbf{1}}$ appears in the i -th component. Then

$$\begin{aligned} \det(nI_k + L + \mathbf{1}) &= \det(r_1 + \vec{\mathbf{1}}, \dots, r_k + \vec{\mathbf{1}}) \\ &= \det(r_1, \dots, r_k) + \sum_{i=1}^k \det(r_1, \dots, \underbrace{\vec{\mathbf{1}}}_i, \dots, r_k) \\ &= (n+k) \frac{p_L(-n)}{n}. \end{aligned}$$

The result follows. \square

Remark 2. To see that Corollary B of [2] is equivalent to the result stated in Part 3, note that in [2], the orders of H_n and $\mathcal{K}(G_n)$ are stated in terms of the characteristic polynomial of the endomorphism of $\ker \delta$ obtained from our Laplacian by restriction. Calling that characteristic polynomial P_G , we have $P_G(x) = p_L(x)/x$.

Remark 3. Part 3 of the theorem also holds in the case $n = 1$, i.e., for the (first) cone over G . The reduced Laplacian of G_1 is $I_k + L$. Therefore, $\mathcal{K}(G_1) \simeq \text{cok}(I_k + L)$, and $|\mathcal{K}(G_1)| = |\det(I_k + L)| = |p_L(-1)|$.

As an immediate corollary of Theorem 1, we have the following:

Corollary 4. ([2, Theorem A]) *There is an exact sequence,*

$$0 \rightarrow (\mathbb{Z}/(n+k)\mathbb{Z})^{n-1} \rightarrow \mathcal{K}(G_n) \rightarrow H_n \rightarrow 0$$

where H_n is a group of order $|p_L(-n)|/n$.

Question 1.2 of [2] asks when the exact sequence in Corollary 4 splits. By Theorem 1, $(\mathbb{Z}/(n+k)\mathbb{Z})^{n-2}$ always splits off of $\mathcal{K}(G_n)$, and the exact sequence of Theorem A splits exactly when $\mathbb{Z}/(n+k)\mathbb{Z}$ is a direct summand of $\text{cok}(nI_k + L + \mathbf{1})$. The latter will depend, for instance, on comparing the prime factorization of $n+k$ to the primary decomposition of the abelian group $\text{cok}(nI_k + L + \mathbf{1})$ (cf. Example 5). It would be interesting if much more could be said in answer to the question for arbitrary G .

Example 5. Let $G = \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$ be the path graph on 4 vertices. For this example, we compute $\mathcal{K}(G_n)$ for all n and show that the exact sequence in Corollary 4 splits if and only if n is odd.

We have

$$M_n := nI_4 + L + \mathbf{1} = \begin{pmatrix} n+2 & 0 & 1 & 1 \\ 0 & n+3 & 0 & 1 \\ 1 & 0 & n+3 & 0 \\ 1 & 1 & 0 & n+2 \end{pmatrix}.$$

If $\text{diag}(d_1, d_2, d_3, d_4)$ is the Smith normal form for M_n , then $\text{cok } M_n \simeq \prod_{i=1}^4 \mathbb{Z}/d_i\mathbb{Z}$. Each d_i may be calculated as the gcd of the $i \times i$ minors of M_n , and $d_i | d_{i+1}$ for $i = 1, 2, 3$. Deleting the second row and third column from M_n produces a 3×3 submatrix matrix with determinant 1. Hence $d_3 = 1$, which forces $d_1 = d_2 = 1$. So M_n is a cyclic group of order $\det(M_n)$. By Theorem 1 (1),

$$\mathcal{K}(G_n) \simeq (\mathbb{Z}/(n+4)\mathbb{Z})^{n-2} \oplus \mathbb{Z}/\det(M_n)\mathbb{Z}.$$

Now

$$\det(M_n) = (n^2 + 4n + 2)(n + 4)(n + 2),$$

and thus $\mathbb{Z}/\det(M_n)\mathbb{Z}$ contains $\mathbb{Z}/(n+4)\mathbb{Z}$ as a direct summand if and only if $n+4$ is relatively prime to $(n^2 + 4n + 2)(n + 2)$. An easy calculation shows that $\gcd((n^2 + 4n + 2)(n + 2), n + 4) = \gcd(n, 4)$, and hence $(\mathbb{Z}/(n+4)\mathbb{Z})^{n-1}$ is a direct summand of $\mathcal{K}(G_n)$ if and only if n is odd.

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